# 7.3: Trigonometric substitution

3 different substitutions we usually use:

- 3. When we see $\sqrt{x^2-1}$ in the integrand, we substitute $x = \sec\theta$ and use identity $\sec^2\theta - 1 = \tan^2\theta$ to get $\sqrt{x^2-1} = \tan\theta$.
- 3'. (We will about it on Wednesday, Sept 12) When we see $\sqrt{x^2-a^2}$ in the integrand, we substitute $x = a\sec\theta$ and use identity $\sec^2\theta - 1 = \tan^2\theta$ to get $\sqrt{x^2-a^2} = a\tan\theta$.

# 7.3: Trigonometric substitution

- Example 3. $\int \frac{dx}{\sqrt{x^2-3}}$.
  Let $x = \sqrt{3}\sec t$. Then $\sqrt{x^2-3} = \sqrt{3}\tan t$.

$$\int \frac{dx}{\sqrt{x^2-3}} = \int \frac{\sqrt{3}\tan t \sec t}{\sqrt{3}\tan t} dt = \int \sec t dt \tag{29}$$

Again, we use the fact $\int \sec t dt = \ln|\sec t + \tan t| + C$. (For a proof, read p483 on textbook.)

# 7.3: Trigonometric substitution

How to write it in terms of $x$, using $x = \sqrt{3}\sec t$?

Draw a picture.

$$\int \frac{dx}{\sqrt{x^2-3}} = \ln(x + \sqrt{x^2-3}) + C.$$

# 7.3: Trigonometric substitution

- Remark we can do $\int \frac{dx}{\sqrt{x^2+3}}$ in a similar way .
  Let $x = \sqrt{3}\tan t$. Then $\sqrt{x^2+3} = \sqrt{3}\sec t$.

$$\begin{aligned} &\int \frac{dx}{\sqrt{x^2+3}} \\ =&\int \frac{\sqrt{3}\sec^2 t}{\sqrt{3}\sec t}dt \\ =&\int \sec t dt \end{aligned} \tag{30}$$

Again, we use the fact $\int \sec t dt = \ln|\sec t + \tan t| + C$. (For a proof, read p483 on textbook.)

# 7.3: Trigonometric substitution

Substitute back to $x$, we have

$$\int \frac{dx}{\sqrt{x^2+3}} = \ln(x + \sqrt{x^2+3}) + C.$$

# 7.3: Trigonometric substitution

- Example 4. $\int \sqrt{1+4x-2x^2}dx$.

  Complete the square first.

  $1+4x-2x^2 = -2(x-1)^2+3$. Thus

$$\begin{aligned} &\int \sqrt{1+4x-2x^2} \\ =&\int \sqrt{3-2(x-1)^2}dx \\ =&\int \sqrt{2}\sqrt{\frac{3}{2}-t^2}dt \end{aligned} \tag{31}$$

  where in the last equality, we substitute $t = x-1$.

# 7.3: Trigonometric substitution

We now use substitution (of case 1') $t = \sqrt{\frac{3}{2}} \sin\theta$

$$\begin{aligned}
&\int \sqrt{2}\sqrt{\frac{3}{2} - t^2}dt \\
=&\int \sqrt{2}\sqrt{\frac{3}{2} - \frac{3}{2}\sin^2\theta}\sqrt{\frac{3}{2}}\cos\theta d\theta \\
=&\frac{3\sqrt{2}}{2}\int \cos^2\theta d\theta \\
=&\frac{3\sqrt{2}}{2}\int \frac{1}{2}(1+\cos 2\theta)d\theta \\
=&\frac{3\sqrt{2}}{4}(\theta + \frac{\sin 2\theta}{2}) + C
\end{aligned} \tag{32}$$

# 7.3: Trigonometric substitution

We now write the answer in terms of the original variable $x$. Using $\sin\theta = \frac{\sqrt{2}t}{\sqrt{3}}$, draw the picture of the right triangle. Thus $\cos\theta = \sqrt{1 - \frac{2t^2}{3}}$. Therefore $\sin 2\theta = 2\sin\theta\cos\theta = 2\frac{\sqrt{2}t}{\sqrt{3}}\sqrt{1-\frac{2t^2}{3}}$. Thus

$$\begin{aligned}
&\frac{3\sqrt{2}}{4}(\theta + \frac{\sin 2\theta}{2}) + C \\
=&\frac{3\sqrt{2}}{4}\left[\arcsin(\sqrt{\frac{2}{3}}t) + \frac{\sqrt{2}t}{\sqrt{3}}\sqrt{1-\frac{2t^2}{3}}\right] + C \\
=&\frac{3\sqrt{2}}{4}\left[\arcsin(\sqrt{\frac{2}{3}}(x-1)) + \frac{\sqrt{2}(x-1)}{\sqrt{3}}\sqrt{1-\frac{2(x-1)^2}{3}}\right] + C.
\end{aligned} \tag{33}$$

# 7.3: Trigonometric substitution

■ Example 5. Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Notice that the ellipse is symmetric with respect to both axes. So it is enough to compute the area in the 1st quadrant, where $x \geq 0$, $y \geq 0$.

$$y = \frac{b}{a}\sqrt{a^2 - x^2}, \quad \text{for} \quad y \geq 0.$$

# 7.3: Trigonometric substitution

Then $\frac{1}{4}$ of the total area $A$ equals

$$\frac{1}{4}A = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}dx.$$

Let $x = a\sin\theta$, then $dx = \cos\theta d\theta$. When $x = 0, \sin\theta = 0$, so $\theta = 0$. $x = a, \sin\theta = 1$, so $\theta = \frac{\pi}{2}$.

# 7.3: Trigonometric substitution

Therefore,

$$\begin{aligned}\frac{1}{4}A =& \frac{b}{a}\int_0^a \sqrt{a^2-x^2}dx \\ =& \frac{b}{a}\int_0^{\frac{\pi}{2}} a\cos\theta\cdot a\cos\theta d\theta \\ =& ab\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta \\ =& ab\int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2\theta)d\theta \\ =& \frac{1}{4}\pi ab.\end{aligned} \tag{34}$$

Thus $A = \pi ab$.