Remark we can do \( \int \frac{dx}{\sqrt{x^2+3}} \) in a similar way.

Let \( x = \sqrt{3} \tan t \). Then \( \sqrt{x^2+3} = \sqrt{3} \sec t \).

\[
\int \frac{dx}{\sqrt{x^2+3}} = \int \frac{\sqrt{3} \sec^2 t}{\sqrt{3} \sec t} dt = \int \sec t dt \tag{26}
\]

Again, we use the fact \( \int \sec t dt = \ln |\sec t + \tan t| + C \). (For a proof, read p483 on textbook.)
7.3: Trigonometric substitution

Substitute back to $x$, we have

$$\int \frac{dx}{\sqrt{x^2 + 3}} = \ln(x + \sqrt{x^2 + 3}) + C.$$
Example 4. Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Notice that the ellipse is symmetric with respect to both axes. So it is enough to compute the area in the 1st quadrant, where $x \geq 0$, $y \geq 0$.

$$y = \frac{b}{a} \sqrt{a^2 - x^2}, \quad \text{for} \quad y \geq 0.$$
Then \( \frac{1}{4} \) of the total area \( A \) equals

\[
\frac{1}{4} A = \int_{0}^{a} \frac{b}{a} \sqrt{a^2 - x^2} \, dx.
\]

Let \( x = a \sin \theta \), then \( dx = \cos \theta \, d\theta \). When \( x = 0 \), \( \sin \theta = 0 \), so \( \theta = 0 \). \( x = a \), \( \sin \theta = 1 \), so \( \theta = \frac{\pi}{2} \).
7.3: Trigonometric substitution

Therefore,

\[
\frac{1}{4} A = \frac{b}{a} \int_{0}^{a} \sqrt{a^2 - x^2} \, dx
\]

\[
= \frac{b}{a} \int_{0}^{\frac{\pi}{2}} a \cos \theta \cdot a \cos \theta \, d\theta
\]

\[
= ab \int_{0}^{\frac{\pi}{2}} \cos^2 \theta \, d\theta
\]

\[
= ab \int_{0}^{\frac{\pi}{2}} \frac{1}{2} (1 + \cos 2\theta) \, d\theta
\]

\[
= \frac{1}{4} \pi ab.
\]

Thus \( A = \pi ab \).
Example 5.

\[
\int_{\sqrt{3}}^{2\sqrt{3}} \frac{1}{x^2\sqrt{x^2 + 4}} \, dx
\]

Let \( x = 2\tan \theta, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \).

\( x = \frac{2}{\sqrt{3}}, \tan \theta = \frac{1}{\sqrt{3}}. \) Thus \( \theta = \frac{\pi}{6} \).

\( x = 2\sqrt{3}, \tan \theta = \sqrt{3}. \) Thus \( \theta = \frac{\pi}{3} \). Then \( dx = 2\sec^2 \theta \, d\theta \). Thus \( \sqrt{x^2 + 4} = \sqrt{4(\tan^2 \theta + 1)} = \sqrt{4\sec^2 \theta} = 2\sec \theta \). Hence

\[
\int_{\sqrt{3}}^{2\sqrt{3}} \frac{1}{x^2\sqrt{x^2 + 4}} \, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{2\sec^2 \theta}{4\tan^2 \theta 2\sec \theta} \, d\theta
\]

\[
= \frac{1}{4} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec \theta}{\tan^2 \theta} \, d\theta
\]

\[
= \frac{1}{4} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos \theta}{\sin^2 \theta} \, d\theta
\]
7.3: Trigonometric substitution

\[
\int_{\pi/6}^{\pi/3} \frac{1}{\sin^2 \theta} d\sin \theta = - \frac{1}{4} \left[ \frac{1}{\sin \theta} \right]_{\pi/6}^{\pi/3} = - \frac{1}{2\sqrt{3}} + \frac{1}{2}. \tag{29}
\]
Example 6

\[
\int \sqrt{x^2 + 2x + 3} \, dx = \int \sqrt{(x + 1)^2 + 2} \, dx = \int \sqrt{t^2 + 2} \, dt
\]

This reduces the problem to something we are familiar with. Take \( t = \sqrt{2} \tan \theta \), just like before.
\[
\frac{1}{4} A = \int_0^a \sqrt{1 - \frac{x^2}{a^2}} \cdot b^2 \, dx
\]
\[
A = 4 \cdot \int_0^a \sqrt{1 - \frac{x^2}{a^2}} \cdot b^2 \, dx
\]
\[
= 4 \cdot b \int_0^a \sqrt{1 - \frac{x^2}{a^2}} \, dx
\]
\[
= 4 \cdot b \int_0^a \sqrt{1 - \sin^2 \theta} \cdot a \cos \theta \, d\theta
\]
\[
= 4b \int_0^{\frac{\pi}{2}} a \cos^2 \theta \, d\theta
\]

\[
x = a \sin \theta
\]
\[
x = a, \quad \sin \theta = 1, \quad \theta = \frac{\pi}{2}
\]
\[
x = 0, \quad \sin \theta = 0, \quad \theta = 0
\]

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{for} \quad a \geq b
\]
\[
x = a, \quad y = 0
\]
\[
y = f(x)
\]
\[
x = a \quad x = b
\]
\[
\text{Area} = \int_a^b f(x) \, dx
\]
\[ = 4b \int_0^{\pi/2} a \cos^2 \theta \, d\theta \]
\[ = 2ba \int_0^{\pi/2} (\cos^2 \theta + 1) \, d\theta \]
\[ = 2ba \left[ \frac{1}{2} \sin 2\theta \right]_0^{\pi/2} + 0 \left[ \frac{\pi}{2} \right] \]
\[ = 2ba \left[ 0 + \frac{\pi}{2} \right] = ab \pi \]

Check: \( a = b, \) circle, \( \pi a^2 \) \( \checkmark \)

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \pi \cdot a^2 = \text{Area} \]
\[
\frac{1}{4} \int \frac{\pi}{3} \frac{1}{\sin \Theta} \, d\Theta = \frac{1}{2} \sin \Theta
\]

\[
= \frac{1}{4} \int \frac{d\sqrt{3}}{\alpha} \frac{1}{u^2} \, du
\]

\[
= -\frac{1}{4} \frac{1}{u} \left[ \frac{\sqrt{3}}{2} \right]_{u=\frac{1}{2}}^{u=\frac{\sqrt{3}}{2}} = -\frac{1}{4} \left( \frac{2}{\sqrt{3}} - 2 \right)
\]
\[
\int \int \sqrt{-x^2 + 3x + 1} \, dx
\]

\[
= \int \sqrt{-(x - \frac{3}{2})^2 + \frac{13}{4}} \, dx
\]

\[
= \int \int \sqrt{-t^2 + \frac{13}{4}} \, dt
\]

\[
x - \frac{3}{2} = t \implies \sqrt{a^2 + t^2}
\]

In the form \( \sqrt{a^2 + t^2} \)

\[
t = \sqrt{\frac{13}{4}} \sin \theta \text{ is the correct trig substitution}
\]

\[
-x^2 + 3x + 1
\]

\[
= -\left( x + a \right)^2 + b
\]

\[
= -\left( x^2 + 2ax + a^2 \right) + b
\]

\[
-2a = 3 \implies a = -\frac{3}{2}
\]

\[
-a^2 + b = 1 \implies b = \frac{9}{4} + 1
\]

\[
= \frac{13}{4}
\]