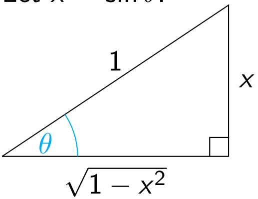
7.3: Trigonometric substitution

- We know $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x$.

  How did we show it in Calc I?

  Let $x = \sin\theta$.



  Then $\sqrt{1-x^2} = \cos\theta$ and $dx = \cos\theta d\theta$.

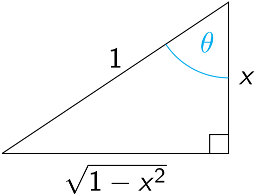
$$\begin{aligned} \int \frac{1}{\sqrt{1-x^2}} dx &= \int d\theta \\ &= \theta + C \\ &= \arcsin x + C. \end{aligned} \tag{24}$$

7.3: Trigonometric substitution

- Using a similar idea of substitution, we can handle more complicated situation.

  Example 1. $\int \frac{\sqrt{1-x^2}}{x^2} dx$.

  Let $x = \cos\theta$.



7.3: Trigonometric substitution

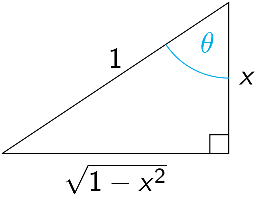
Then $\sqrt{1-x^2} = \sin\theta$ and $dx = -\sin\theta d\theta$.

$$\begin{aligned}\int \frac{\sqrt{1-x^2}}{x^2} dx &= \int \frac{\sin\theta}{\cos^2\theta}(-\sin\theta) d\theta \\ &= -\int \tan^2\theta d\theta \\ &= -\int (\sec^2\theta - 1) d\theta \\ &= -\tan\theta + \theta + C.\end{aligned} \tag{25}$$

7.3: Trigonometric substitution

How to write it in terms of x, using $x = \cos\theta$?

See the previous picture:



From this picture,

$\tan\theta = \frac{\sqrt{1-x^2}}{x}$. Thus

$$-\tan\theta + \theta + C = -\frac{\sqrt{1-x^2}}{x} + \arccos x + C.$$

7.3: Trigonometric substitution

- The method to write $\sqrt{1-x^2} = \sin\theta$ in Example 1 is called Trigonometric substitution. It is extremely effective when dealing with square root of a quadratic function, because after using the trig substitution, the argument of the square root is a "perfect square". Thus we can "get rid" of the radical symbol.
- Similarly, we substitute $x = \tan\theta$ if there is a term $\sqrt{1+x^2}$. Since $1+\tan^2\theta = \sec^2\theta$,

$$\sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = \sec\theta.$$

7.3: Trigonometric substitution

3 different substitutions we usually use:

- 1. When we see $\sqrt{1-x^2}$ in the integrand, we substitute $x=\sin\theta$ (or $x=\cos\theta$) and use Pythagorean identity to get $\sqrt{1-x^2}=\sqrt{1-\sin^2\theta}=\cos\theta$ (or $\sqrt{1-x^2}=\sqrt{1-\cos^2\theta}=\sin\theta$) .
- 2. When we see $\sqrt{1+x^2}$ in the integrand, we substitute $x=\tan\theta$ and use identity $1+\tan^2\theta=\sec^2\theta$ to get $\sqrt{1+x^2}=\sec\theta$.
- 3. (We will about it on Wednesday, Sept 12) When we see $\sqrt{x^2-1}$ in the integrand, we substitute $x=\sec\theta$ and use identity $\sec^2\theta-1=\tan^2\theta$ to get $\sqrt{x^2-1}=\tan\theta$.

7.3: Trigonometric substitution

The variants of above trig substitution

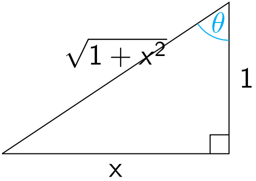
- 1'. When we see $\sqrt{a^2 - x^2}$ in the integrand, we substitute $x = a\sin\theta$ (or $x = a\cos\theta$) and use Pythagorean identity to get $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\cos\theta$ (or $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\cos^2\theta} = a\sin\theta$) .
- 2'. When we see $\sqrt{a^2 + x^2}$ in the integrand, we substitute $x = a\tan\theta$ and use identity $1 + \tan^2\theta = \sec^2\theta$ to get $\sqrt{a^2 + x^2} = a\sec\theta$.
- 3'. (We will about it on Wednesday, Sept 12) When we see $\sqrt{x^2 - a^2}$ in the integrand, we substitute $x = a\sec\theta$ and use identity $\sec^2\theta - 1 = \tan^2\theta$ to get $\sqrt{x^2 - a^2} = a\tan\theta$.

7.3: Trigonometric substitution

- An example of 2.

  Example 2. $\int \sqrt{1+x^2}dx$.

  Let $x = \tan\theta$.



7.3: Trigonometric substitution

Then $\sqrt{1+x^2} = \sec\theta$ and $dx = \sec^2\theta d\theta$.

$$\int \sqrt{1+x^2}dx = \int \sec\theta(\sec^2\theta)d\theta = \int \sec^3\theta d\theta \qquad (26)$$

$$\begin{aligned}\int \sec^3\theta d\theta &\overset{Using IBP}{=} \sec\theta\tan\theta - \int \tan\theta(\sec\theta)' d\theta \\ &= \sec\theta\tan\theta - \int \tan^2\theta\sec\theta d\theta \\ &= \sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta d\theta \\ &= \sec\theta\tan\theta - \int \sec^3\theta d\theta + \ln|\sec\theta + \tan\theta| + C.\end{aligned} \qquad (27)$$

7.3: Trigonometric substitution

Here we have used the fact $\int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C$. For a proof, read p483 on textbook.

Thus we get

$$2\int \sec^3\theta d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C. \tag{28}$$

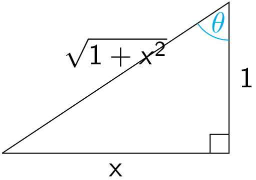
Plugging it into (23), the final answer is that

$$\int \sqrt{1+x^2}dx = \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) + C$$

7.3: Trigonometric substitution

How to write it in terms of x, using $x = \tan\theta$?

Use the previous picture.



Thus $\sec\theta = \frac{1}{\cos\theta} = \sqrt{1+x^2}$. We also have $\tan\theta = x$. So

$$\frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) + C = \frac{1}{2}(x\sqrt{1+x^2} + \ln|\sqrt{1+x^2} + x|) + C.$$