AS.110.109: Calculus II (Eng)

Review session Midterm 1

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7.1: Integration by parts

- Basic integration method: u-sub, integration table
- **Integration By Parts formula**

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

General Principle

We learn how to do integration by parts from experiences.

- Priority order to choose f:
  - inverse (e.g. $\arcsin x$, $\arccos x$, etc.)
  - logarithmic (e.g. $\log x$, $\log_2 x$, $\log_{10} x$ etc.)
  - algebraic (e.g. x^3, x^9, etc.)
  - trigonometric (e.g. $\sin x$, $\cos x$, etc.)
  - exponential (e.g. $e^x, 2^x, 3^x$, etc.)

ILATE

7.2: Integrals of trigonometrics I

- Goal: evaluate $\int \sin^m x \cos^n x dx$, where m, n are integers.
- Strategy:
  - Case 1. n is an odd integer
  - Case 2. m is an odd integer
  - Case 3. Both m and n are even, we use half-angle identity.

7.2: Integrals of trigonometrics II

- Goal: evaluate $\int \tan^m x \sec^n x dx$, where m, n are integers.
- Strategy:
  - Case 1. If n is an even integer ($n = 2k$), then save a factor $\sec^2 x$ in the integrand.

    Key fact:

$$(\tan x)' = \sec^2 x, \quad \text{and} \quad \sec^2 x = 1 + \tan^2 x.$$

  - Case 2. If m is an odd integer ($m = 2k + 1$), then save a factor $\sec x \tan x$ in the integrand.

    Key fact:

$$(\sec x)' = \sec x \tan x, \quad \text{and} \quad \sec^2 x = 1 + \tan^2 x.$$

7.3: Trigonometric substitution

3 different substitutions we usually use:

- $\sqrt{a^2 - x^2} = a \sin t$ by substituting $x = a \cos t$

and using identity $1 - \cos^2 t = \sin^2 t$.

- $\sqrt{a^2 + x^2} = a \sec t$ by substituting $x = a \tan t$

and using identity $1 + \tan^2 t = \sec^2 t$.

- $\sqrt{x^2 - a^2} = a \tan t$ by substituting $x = a \sec t$

and using identity $\sec^2 t - 1 = \tan^2 t$.

7.4: Integration of rational functions

A rational function is a function of the form:

$$f(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials in x.

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0.$$

$$Q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0.$$

How to express a rational function as a linear combination of partial fractions?

7.4: Integration of rational functions

Definition

$f(x) = \frac{P(x)}{Q(x)}$ is called proper if $deg(P(x)) < deg(Q(x))$.
Otherwise, it is called improper.

A rational function can always be written as:

$$f(x) = \text{a polynomial function} + \text{a proper rational function.}$$

7.4: Integration of rational functions

After reducing the rational function to a proper rational function. We need to factorize $Q(x)$ to irreducible polynomials.

- All degree 1 polynomials are irreducible. Example. $x+3$, $2x+4$, etc.
- Some degree 2 polynomials are irreducible, but not all of them.
  - if the discriminant $\Delta := b^2 - 4ac \geq 0$, then reducible;
  - if the discriminant $\Delta := b^2 - 4ac < 0$, then irreducible.

7.4: Integration of rational functions

If $Q(x)$ has degree≥ 3, it can always be factored into a product:

$$Q(x) = Q_1(x) \cdots Q_k(x),$$

where each $Q_i(x)$ is a irreducible polynomial with degree ≤ 2.

Q_i is either of degree 1, or of degree 2 and $\Delta < 0$.

7.4: Integration of rational functions

To integrate a rational function, we factorize the denominator $Q(x)$:

$$Q(x) = Q_1(x) \cdots Q_k(x).$$

Case 1. $Q(x)$ is a product of distinct linear factors. $\frac{P(x)}{Q(x)}$ takes the form

$$\frac{A_1}{a_1 x + b_1} + \cdots + \frac{A_k}{a_k x + b_k}$$

.

- Example 7. $\int \frac{x}{(x+1)(x+2)} dx$

$$\frac{x}{(x+1)(x+2)} = \frac{A_1}{x+1} + \frac{A_2}{x+2}.$$

7.4: Integration of rational functions

Case 2. $Q(x)$ is a product of linear factors, some of which are repeated.

Suppose $Q(x)$ contains a term $(a_1x+b_1)^{r_1}$. Instead of $\frac{A_1}{a_1x+b_1}$, we would have

$$\frac{A_1}{a_1x+b_1}+\cdots+\frac{A_{r_1}}{(a_1x+b_1)^{r_1}}.$$

For another repeated factor, say $(a_2x+b_2)^{r_2}$, we have

$$\frac{B_1}{a_2x+b_2}+\cdots+\frac{B_{r_2}}{(a_2x+b_2)^{r_2}}.$$

7.4: Integration of rational functions

Case 3. $Q(x)$ has irreducible quadratic factors $ax^2 + bx + c$, none of which is repeated. $\frac{P(x)}{Q(x)}$ takes the form $\frac{Ax+B}{ax^2+bx+c}$.

- Example 9. $\int \frac{x^2}{(x+1)(x^2+1)} dx$

$$\frac{x^2}{(x+1)(x^2+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}.$$

7.4: Integration of rational functions

Case 4. $Q(x)$ has irreducible quadratic factors $ax^2 + bx + c$, some of which is repeated. For example. if there is a factor $\int \frac{1}{(x^2+a^2)^r} dx$ in $Q(x)$. Then in the decomposition, there are following terms in $\frac{P(x)}{Q(x)}$: (of course, there are other terms corresponding to other factors of Q.)

$$\int \frac{A_1 x + B_1}{(x^2 + a^2)^1} dx + \cdots + \frac{A_r x + B_r}{(x^2 + a^2)^r} dx. \tag{1}$$

Integration of $\int \frac{A_r x}{(x^2+a^2)^r} dx$ uses u sub; that of $\int \frac{B_r}{(x^2+a^2)^r} dx$ may need to use trig substitution.

Differential equations

- An equation involves derivatives of a function is called a differential equation.
- Solution is a function $y = y(x)$.
- If besides the differential equation, the value of y at a specific point, say x_0 is given: $y(x_0) = y_0$ is given, we call it an initial value problem.

Geometric meaning of general solution

Each solution is a curve on the xy-plane.

The set of general solution is a family of (infinitely many) curves on the xy-plane.

Geometrically, the initial condition has the effect of isolating the integral curve that passes through the point (x_0, y_0).

9.2: Direction fields and Euler method

For $y' = F(x, y)$, direction fields is a vector $F(x_0, y_0)$ at each point (x_0, y_0).

9.3: Separable Equations

An equation is separable if one can write it as $y'(x) = f(x)g(y)$.

Method: use algebra to separate the two variables, so that each side of the equation only contains 1 variable.

Example

Example. Find all the solutions to

$$y' = 2x(1-y)^2.$$

Solution: First note that there is a constant solution $y \equiv 1$.

Next we use separation method as above

$$\Rightarrow \frac{dy}{(1-y)^2} = 2xdx.$$

Then integrate both sides,

$$\Rightarrow \frac{1}{1-y} = x^2 + C.$$

$$\Rightarrow y = 1 - \frac{1}{x^2+C}.$$

This together with $y \equiv 1$ are the general solutions.

Orthogonal trajectories.

Find orthogonal trajectories.

Example: Take a family of curves $x = ky^2$, where k is any constant. Find another family of curves such that any member of this family intersects any given one at a right angle.

$$x = ky^2$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{2ky} = \frac{y}{2x}.$$

The orthogonal trajectories satisfy the differential equation:

$$\frac{dy}{dx} = -\frac{2x}{y}.$$
$$\Rightarrow \int y dy = -\int 2x dx.$$
$$\Rightarrow x^2 + \frac{y^2}{2} = C, \quad C > 0.$$

Linear equations

A first order linear differential equation is

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Linear equations

Linear equations $\frac{dy}{dx} + P(x)y = Q(x)$:

- Step 1. Take integrating factor $I(x) = e^{\int P(x)dx}$, and multiply it on both sides to get

$$(I(x)y(x))' = I(x)Q(x)$$

- Step 2. Integrate both sides to get the solution

$$y(x) = \frac{1}{I(x)} \int I(x)Q(x)dx.$$

Parametric curves

Parametric equations of curve takes the form

$$x = f(t), y = g(t),$$

where $t \in (a, b)$.

The tangent slope

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}},$$

if $\frac{dx}{dt} \neq 0$.

Calculus with parametric curves

Tangent line equation at $t = t_0$ is given by

$$y(x) = slope \cdot (x - x(t_0)) + y(t_0).$$

- At points where $\frac{dy}{dx} = 0$, the tangent line is horizontal.
- At points where $\frac{dy}{dx} = \pm\infty$, the tangent line is vertical.

Calculus with parametric curves

- Second order derivative is

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$