Calculus with parametric curves

- Area under a parametric curve $x = x(t), y = y(t)$.

$$Area = \int_a^b y(x)dx.$$

Plug $x = x(t), y = y(t)$ into it,

$$Area = \int_{t_1}^{t_2} y(t)x'(t)dt.$$

t_1 is chosen such that $x(t_1) = a$; t_2 is chosen such that $x(t_2) = b$.

Note $x(t_1) < x(t_2)$!

Calculus with parametric curves

- Example 3. Area under the half circle $x = 2\cos t$, $y = 2\sin t$, $t \in [0, \pi]$.
  Solution: Draw the picture, $a = -2, b = 2$. Thus $t_1 = \pi$, $t_2 = 0$.
  (t_1, t_2 are chosen such that $x(t_1) < x(t_2)$!)

Calculus with parametric curves

Then

$$Area = \int_{t_1}^{t_2} y(t)x'(t)dt = \int_{\pi}^{0} -4\sin^2 t dt$$

Since $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, we have

$$Area = \int_0^{\pi} 4 \cdot \frac{1-\cos 2t}{2} dt = 2\pi.$$

Calculus with parametric curves

- Arc length for the curve given by the graph of function $y = y(x)$

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx,$$

- Arc length for parametric curve $x = x(t), y = y(t),\ t \in [t_1, t_2]$

$$L = \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2} dt. \tag{4}$$

Note that to compute the length, we need to choose $t_1 < t_2$. Only this way, the integral is positive.

Calculus with parametric curves

- Example 4. Arc length of $x = \cos t$, $y = \sin t$, $t \in [0, \pi/3]$.

  Solution: $x'(t) = (\cos t)' = -\sin t$, $y'(t) = (\sin t)' = \cos t$,

$$
\begin{aligned}
L &= \int_{t_1}^{t_2} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt \\
&= \int_0^{\pi/3} \sqrt{(-\sin t)^2 + \cos^2 t} dt \\
&= \int_0^{\pi/3} 1 dt = \pi/3.
\end{aligned}
\tag{5}
$$