Series

Definition

$a_1 + a_2 + a_3 + \cdots$ is called an infinite series or just series.

Denoted by

$$\sum_{n=1}^{\infty} a_n, or \sum a_n.$$

Series

Given a series $\sum_{n=1}^{\infty} a_n$. The partial sum is the sum of the first n terms of the series, denoted by s_n.

$$s_n := \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

If $\lim_{n\to\infty} s_n$ exists as a finite number, then the series

$$\sum_{n=1}^{\infty} a_n := \lim_{n\to\infty} s_n,$$

and we say it is convergent.

If $\lim_{n\to\infty} s_n$ does not exist, we say $\sum_{n=1}^{\infty} a_n$ is divergent.

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- Example 1. Suppose $s_n = \frac{3n}{2n+3}$. Then by definition, $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{3n}{2n+3} = \frac{3}{2}$.

Series

- Example 2. Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent. And it is equal to 1.

  Solution: The partial sum

$$s_n = \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n(n+1)}.$$

  Note that $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$. So

$$s_n = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) \cdots + (\frac{1}{n} - \frac{1}{n+1})$$

  All the terms, except 1 and $-\frac{1}{n+1}$, cancel. So $s_n = 1 - \frac{1}{n+1}$.

  Hence

$$\lim_{n\to\infty} s_n = 1.$$

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By definition, it means $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and it is equal to 1.

- Very few series have such a beautiful cancellation formula. For most of the series, we cannot compute the partial sum explicitly.

Series

- Example 3. (Geometric series) Compute $\sum_{n=0}^{\infty} a \cdot r^n$. For what values of r, this series converges.

  ▶ Remark: Recall $\{a_n\}$ is a geometric sequence if $\frac{a_{n+1}}{a_n}$ equals a constant r, for all n.

  Solution: If $r \neq 1$,

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{1 - r^n}{1 - r}. \tag{1}$$

  When $r = 1$, $s_n = a + a + \cdots a = na$.

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How to prove identity (1)?

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$r \cdot s_n = ar + ar^2 + ar^3 + \cdots + ar^n$$

Subtract the first line by the second line gives

$$(1 - r)s_n = a - ar^n$$

which is equivalent to

$$s_n = a\frac{1 - r^n}{1 - r}$$

for $r \neq 1$.

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If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$. Thus

$$\lim_{n\to\infty} s_n = \frac{a}{1-r}.$$

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If $r = 1$, then the partial sum s_n is not equal to $a\frac{1-r^n}{1-r}$. It is $s_n = na$, whose limit is infinity.
If $r \leq -1$ or $r > 1$, then limit of r^n does not exist. (Recall the four cases in Chapter 11.1, in Oct 29 notes.) Hence limit of s_n does not exist.

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Conclusion: If $-1 < r < 1$, then the series converges and

$$a + ar + ar^2 + \cdots = \sum_{n=0}^{\infty} ar^n = a\frac{1}{1-r}.$$

If $r \leq -1$ or $r \geq 1$, then $\sum_{n=0}^{\infty} ar^n$ diverges.

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Example 4. Find the sum of geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots .$$

Solution: Note $\frac{a_{n+1}}{a_n}$ is a constant for all n, and they all equal to $-\frac{2}{3}$. Thus it is a geometric series. $a = 5$, $r = \frac{a_{n+1}}{a_n} = \frac{a_2}{a_1} = -\frac{2}{3}$.

Apparently, $|r| < 1$.

Thus the series equals to

$$a\frac{1}{1-r} = 5 \cdot \frac{1}{1-(-\frac{2}{3})} = \frac{5}{1+\frac{2}{3}} = 3.$$

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- Example 5. Is the series $\sum_{n=1}^{\infty} 5^{2n}2^{1-n}$ convergent or divergent?

  Solution:

$$\sum_{n=1}^{\infty} 5^{2n}2^{1-n} = \sum_{n=1}^{\infty} (5^2)^n \cdot 2 \cdot \frac{1}{2^n} = \sum_{n=1}^{\infty} 2 \cdot (\frac{25}{2})^n.$$

  Note $r = \frac{25}{2} > 1$, thus the geometric series diverges.

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- Example 6. Find $\sum_{n=100}^{\infty} 2^{-n}$.

  Solution:

$$\begin{aligned}\sum_{n=100}^{\infty} 2^{-n} =&(\frac{1}{2})^{100} + (\frac{1}{2})^{101} + (\frac{1}{2})^{102} + (\frac{1}{2})^{103} + \cdots \\ =&(\frac{1}{2})^{100}\left(1 + \frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \cdots\right) \qquad (2) \\ =&(\frac{1}{2})^{100}\sum_{n=0}^{\infty}\frac{1}{2^n}.\end{aligned}$$

$|r| = \frac{1}{2} < 1$, thus $\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2$.

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Hence

$$\sum_{n=100}^{\infty} 2^{-n} = (\frac{1}{2})^{100} \cdot \frac{1}{1-\frac{1}{2}} = (\frac{1}{2})^{99}.$$

This example shows that we can compute geometric series starting with any index.

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Theorem

If $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$*.*

- Remark: The converse is not true in general. If $\lim_{n\to\infty} a_n = 0$, the series $\sum_{n=1}^{\infty} a_n$ may be convergent and divergent. We will give examples in future.

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Contrapositive Statement of the Theorem: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

This is also called Divergence test.

- Example 7. $\sum_{n=1}^{\infty} \frac{n}{2n+4}$ diverges.

  Solution: Since $\lim_{n\to\infty} \frac{n}{2n+4} = \frac{1}{2} \neq 0$, by the contrapositive statement of the Theorem, $\sum_{n=1}^{\infty} \frac{n}{2n+4}$ diverges.