Calculus with parametric curves

- Describing the tangents of curves. Suppose

$$x = x(t), y = y(t) \tag{1}$$

is a parametric curve. What is the tangent direction of the curve at time t?

Solution: The tangent slope is given by $\frac{dy}{dx}$ if y is a function of x. Using the chain rule,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}},$$

if $\frac{dx}{dt} \neq 0$.

Calculus with parametric curves

- Example 1. Parametric equation of the cycloid is given by

$$x = \theta - \sin\theta, y = 1 - \cos\theta. \tag{2}$$

Find the tangent where $\theta = \frac{\pi}{3}$.

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Solution: Use the chain rule,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}.$$

Since $\frac{dy}{d\theta} = \sin\theta$, and $\frac{dx}{d\theta} = 1 - \cos\theta$, we have

$$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}.$$

So at $\theta = \frac{\pi}{3}$, thus $\frac{dy}{dx} = \sqrt{3}$.

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Tangent line equation:

$$y(x) - y_0 = slope \cdot (x - x_0).$$

At $\theta = \frac{\pi}{3}$,

$$x_0 = \frac{\pi}{3} - \frac{\sqrt{3}}{2}, y_0 = \frac{1}{2}.$$

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Tangent line equation of cycloid at $\theta = \frac{\pi}{3}$ is given by:

$$y(x) = \sqrt{3} \cdot [x - (\frac{\pi}{3} - \frac{\sqrt{3}}{2})] + \frac{1}{2}.$$

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- At what points (if any), the tangent line is horizontal or vertical?
  - At points where $\frac{dy}{dx} = 0$, the tangent line is horizontal.

Since

$$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta},$$

this happens if and only if $\sin\theta = 0$, and $1 - \cos\theta \neq 0$. Namely, $\theta = (2n-1)\pi$, for all integer n.

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- At points where $\frac{dy}{dx} = \pm\infty$, the tangent line is vertical.

Since

$$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta},$$

this happens if and only if $1-\cos\theta = 0$. Namely, $\theta = 2n\pi$, for all integer n.

Calculus with parametric curves

- Second order derivative $\frac{d^2y}{dx^2}$. Recall that

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Thus

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

- If $\frac{d^2y}{dx^2} > 0$, the curve is convex.
- If $\frac{d^2y}{dx^2} < 0$, the curve is concave.

Calculus with parametric curves

- Example 2. Parametric equation of the cycloid is given by

$$x = \theta - \sin\theta, y = 1 - \cos\theta. \tag{3}$$

Find the value of $\frac{d^2y}{dx^2}$ of the cycloid, and determine if the curve is convex or concave.

Solution:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}}.$$

Calculus with parametric curves

Since we obtain in Example 1 that

$$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta},$$

and

$$\frac{dx}{d\theta} = 1-\cos\theta,$$

we get

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{\sin\theta}{1-\cos\theta}\right)}{1-\cos\theta} = \frac{-1}{(1-\cos\theta)^2}.$$

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Therefore

$$\frac{d^2y}{dx^2} < 0$$

for all θ. Hence the curve is concave everywhere.