Sequences

- Geometric sequence

  $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$.

  $\lim_{n\to\infty} r^n = \infty$ if $|r| > 1$.

  $\lim_{n\to\infty} (-1)^n$ does not exist.

  $\lim_{n\to\infty} 1^n = 1$.

  Conclusion: The sequence $\{r^n\}$ converges if $-1 < r \leq 1$, and it diverges for all other values of r.

Sequences

Definition

If $a_1 > a_2 > a_3 > \cdots$, we say the sequence is decreasing.

If $a_1 < a_2 < a_3 < \cdots$, we say the sequence is increasing.

We say a sequence is monotonic if it is either increasing or decreasing.

Sequences

- Example 8. Show that the sequence $a_n = \frac{n}{n^2+1}$ is decreasing.
  Solution: Note that for $f(x) = \frac{x}{x^2+1}$, $f'(x) = \frac{1-x^2}{(x^2+1)^2}$. It is negative when $x^2 > 1$. Thus f is decreasing. Hence for $n \geq 1$, the sequence is decreasing.

Sequences

Definition

If there is a constant M, so that $a_n \leq M$ for all n, then we say the sequence is bounded from above.

If there is a constant m, so that $m \leq a_n$ for all n, then we say the sequence is bounded from below.

M is an upper bound for $\{a_n\}$. m is a lower bound for $\{a_n\}$.

Sequences

- Fact 5: Every bounded, monotonic sequence is convergent.
- Example 9 (Example 14 on Page 703): Investigate the sequence $\{a_n\}$ defined by the recurrence relation $a_1 = 2$, $a_{n+1} = \frac{1}{2}(a_n + 6)$ for $n = 1, 2, 3, \cdots$.

Sequences

Solution: We compute the first few terms and try to see the pattern.

$a_1 = 2, a_2 = 4, a_3 = 5, a_4 = 5.5, a_5 = 5.75, a_6 = 5.875, ...$

It suggests it is an increasing sequence, and the terms are approaching 6.

Sequences

- Prove the sequence is increasing by induction.

1) $a_2 = 4 > a_1$.

2) Suppose $a_{n+1} > a_n$ is valid for $n = k$, it is also valid for $n = k + 1$.

- Prove the sequence is bounded by 6 by induction.

1) $a_1 = 2 < 6$.

2) Suppose $a_n < 6$ is valid for $n = k$, it is also valid for $n = k + 1$.

Sequences

Therefore $\{a_n\}$ is an increasing and bounded sequence. By Fact 5, it is convergent, i.e. $\lim_{n\to\infty} a_n$ exists.

How to determine $\lim_{n\to\infty} a_n$?(Fact 5 does not determine the value of the limit.)

Denote

$$\lim_{n\to\infty} a_n = L.$$

Using recurrence formula,

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + 6) = \frac{1}{2}(\lim_{n\to\infty} a_n + 6)$

Thus $L = \frac{1}{2}(L + 6)$. This implies $L = 6$.

Series

Definition

$a_1 + a_2 + a_3 + \cdots$ is called an infinite series or just series.

Denoted by

$$\sum_{n=1}^{\infty} a_n, or \sum a_n.$$

Series

Given a series $\sum_{n=1}^{\infty} a_n$, let s_n denote its partial sum

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

If $\lim_{n\to\infty} s_n$ exists as a finite number, then the series

$$\sum_{n=1}^{\infty} a_n := \lim_{n\to\infty} s_n,$$

and we say it is convergent.

If $\lim_{n\to\infty} s_n$ does not exist, we say $\sum_{n=1}^{\infty} a_n$ is divergent.

Series

- Example 1. Suppose $s_n = \frac{3n}{2n+3}$. Then

$$a_n = s_n - s_{n-1} = \frac{3n}{2n+3} - \frac{3(n-1)}{2(n-1)+3} = \frac{3n}{2n+3} - \frac{3n-3}{2n+1}.$$

$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{3n}{2n+3} = \frac{3}{2}$.

Series

- Example 2. Geometric series $a_n = a \cdot r^{n-1}$. Compute $\sum_{n=1}^{\infty} a_n$ for $-1 < r < 1$.
  - Remark: in other words, $\{a_n\}$ is a geometric series if $\frac{a_{n+1}}{a_n}$ equals r for all n.

  Solution:

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{1 - r^n}{1 - r}, \tag{1}$$

  for $r \neq 1$.

Series

How to prove identity (1)?

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$r \cdot s_n = ar + ar^2 + ar^3 + \cdots + ar^n$$

Thus

$$(1 - r)s_n = a - ar^n$$

which is equivalent to

$$s_n = a\frac{1 - r^n}{1 - r}$$

for $r \neq 1$.

Series

If $-1 < r < 1$, then $r^n \to 0$. Thus

$$\lim_{n\to\infty} s_n = a\frac{1}{1-r}.$$

Series

If $r = -1$, then limit of $s_n = a\frac{1-r^n}{1-r}$ does not exist.

If $r = 1$, then the partial sum s_n is not equal to $a\frac{1}{1-r}$. It should be $s_n = na$ whose limit is infinity.

If $r \leq -1$ or $r > 1$, then limit of r^n does not exist. Hence limit of s_n does not exist.

Series

Conclusion: If $-1 < r < 1$, then

$$a + ar + ar^2 + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = a\frac{1}{1-r}.$$

The series converges.

If $r \leq -1$ or $r \geq 1$, then $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

Series

- Example 3. Find the sum of geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots .$$

Solution: Note $\frac{a_{n+1}}{a_n}$ is a constant for all n, and they all equal to $-\frac{2}{3}$. Thus it is a geometric series. $a = 5$, $r = \frac{a_{n+1}}{a_n} = \frac{a_2}{a_1} = -\frac{2}{3}$.

Apparently, $|r| < 1$.

Thus the series equals to

$$a\frac{1}{1-r} = 5 \cdot \frac{1}{1-(-\frac{2}{3})} = \frac{5}{1+\frac{2}{3}} = 3.$$

Series

- Example 4. Is the series $\sum_{n=1}^{\infty} 5^{2n}2^{1-n}$ convergent or divergent?

  Solution:

$$\sum_{n=1}^{\infty} 5^{2n}2^{1-n} = \sum_{n=1}^{\infty}(5^2)^n \cdot 2 \cdot \frac{1}{2^n} = \sum_{n=1}^{\infty} 2 \cdot (\frac{25}{2})^n.$$

  Note $r = \frac{25}{2} > 1$, thus the geometric series diverges.

Series

- Example 5. Find $\sum_{n=2}^{\infty} 2^{-n}$.

  Solution:

$$\sum_{n=2}^{\infty} 2^{-n} = \frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots = \frac{1}{4}(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots) = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}.$$

  Thus

$$\sum_{n=2}^{\infty} 2^{-n} = \frac{1}{4}\cdot\frac{1}{1-\frac{1}{2}} = \frac{1}{2}.$$