Example 7. Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

\[ s_2 = 1 + \frac{1}{2} \]

\[ s_4 = 1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) > 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2} \]

\[ s_8 = s_4 + \left( \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) > 1 + \frac{2}{2} + \frac{1}{2} = 1 + \frac{3}{2} \]

Thus

\[ s_{2^n} > 1 + \frac{n}{2}. \]

Thus $s_n$ diverges. Therefore harmonic series diverges.
Theorem

If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \to \infty} a_n = 0$.

Remark: The converse is not true in general. If $\lim_{n \to \infty} a_n = 0$, the series $\sum_{n=1}^{\infty} a_n$ may be convergent and divergent. In Example 7, $\lim_{n \to \infty} a_n = 0$, but $\sum_{n=1}^{\infty} a_n$ diverges.
Contrapositive Statement of the Theorem: If \( \lim_{n \to \infty} a_n \neq 0 \), then \( \sum_{n=1}^{\infty} a_n \) is divergent.

Example 8. \( \sum_{n=1}^{\infty} \frac{n}{2n+4} \) diverges.

Solution: Since \( \lim_{n \to \infty} \frac{n}{2n+4} = \frac{1}{2} \neq 0 \), by the contrapositive statement of the Theorem, \( \sum_{n=1}^{\infty} \frac{n}{2n+4} \) diverges.
Example 1. Determine if $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent or divergent.

Solution:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

Since $\frac{1}{x^2}$ is decreasing,

$$\frac{1}{2^2} \leq \int_{1}^{2} \frac{1}{x^2} \, dx,$$

$$\frac{1}{3^2} \leq \int_{2}^{3} \frac{1}{x^2} \, dx,$$

$$\frac{1}{4^2} \leq \int_{3}^{4} \frac{1}{x^2} \, dx, \cdots$$
The integral test and estimates of sums

\[ 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \leq 1 + \int_{1}^{2} \frac{1}{x^2} \, dx + \int_{2}^{3} \frac{1}{x^2} \, dx + \int_{3}^{4} \frac{1}{x^2} \, dx + \cdots \]

\[ = 1 + \int_{1}^{\infty} \frac{1}{x^2} \, dx. \]
The integral test and estimates of sums

Since \( \int_{1}^{\infty} \frac{1}{x^2} \, dx \) is convergent, the partial sum \( s_n \) is bounded. Also, \( a_n > 0 \) implies \( s_n \) is increasing.

Recall Fact 5: Every bounded monotonic sequence is convergent.

Thus \( s_n \)'s limit exists, i.e. \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is convergent.
The integral test and estimates of sums

This example shows us that the convergence of $\int_1^\infty \frac{1}{x^2} \, dx$ implies convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

This can be generalized to a large class of series:
The integral test Suppose $f$ is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_{1}^{\infty} f(x) dx$ is convergent.
Example 2. For what values of $p$ is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

Solution: Use the integral test. We know from Chapter 7.8 Indefinite Integral that $\int_1^{\infty} \frac{1}{x^p} \, dx$ is convergent if $p > 1$, and divergent if $p \leq 1$.

Conclusion: The $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$, and divergent if $p \leq 1$. 