AS.110.109: Calculus II (Eng)

Chapter 11: Sequences and Series

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Sequences

- Definition: A sequence (of numbers) is a list of $\{a_1, a_2, a_3, \cdots\}$ ordered by an index set I. I is just the set of positive integers.

  Other ways to write a sequence:

$$\{a_n\}_{n=1}^{\infty},$$

or

$$\{a_n\}_{n\in\mathbb{Z}^+}.$$

Sequences

- The numbers are not necessarily distinct.

  Example 1. $a_1 = 1, a_2 = 0, a_3 = 0, \cdots$ is a sequence.

Sequences

- For some sequences, there is a function $f(x) : \mathbb{Z}^+ \to \mathbb{R}$, such that $a_n = f(n)$.

  Example 2. $a_1 = 1, a_2 = 3, a_3 = 5, a_4 = 7 \cdots$ is a sequence.

  In this example, $f(x) = 2x - 1$. Another way to write it is

$$\{2n - 1\}_{n=1}^{\infty}.$$

Sequences

- Example 3. $a_1 = 2, a_2 = 4, a_3 = 6, a_4 = 8 \cdots$ is a sequence.
  In this example, $f(x) = 2x$. Another way to write it is

$$\{2n\}_{n=1}^{\infty}.$$

Sequences

- Some more examples of sequences.

$$\{n^2\}_{n=1}^{\infty},$$

$$\{\sqrt{n+1}\}_{n=1}^{\infty},$$

$$\{\cos\frac{(n+1)\pi}{2}\}_{n=1}^{\infty},$$

$$\{\frac{1}{n}-\frac{1}{n+1}\}_{n=1}^{\infty}.$$

Sequences

- There are also examples that we cannot write down the general formula of the n-th term in the sequence. Example 4.

  $a_1 = 1, a_2 = 2,$

  $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.

  This is the famous Fibonacci sequence.

Sequences

- There are 3 ways to express a sequence:
  1. list all the terms
  2. a function $f : \mathbb{Z}^+ \to \mathbb{R}$ such that $a_n = f(n)$
  3. an inductive formula.

Sequences

- If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences, then each of the following is a sequence:
  $\{a_{n+1}\}_{n=1}^{\infty}$, $\{a_n^2\}_{n=1}^{\infty}$, $\{a_n + b_n\}_{n=1}^{\infty}$, $\{a_n \cdot b_n\}_{n=1}^{\infty}$, $\{a_n^2 + b_n^2\}_{n=1}^{\infty}$

Sequences

- We will put a lot of effort to understand asymptotic behavior of a sequence, namely as $n \to \infty$.

Definition

If $\lim_{n\to\infty}\{a_n\}$ exists, we say the sequence is convergent.

Otherwise, we say the sequence is divergent.

Sequences

- Property a: Suppose $f(x)$ is an increasing/decreasing function, then $a_n = f(n)$ is an increasing sequence/decreasing sequence for all $n \geq 1$.
- Property b: Suppose $f(x)$ is a function so that $a_n = f(n)$ for all $n \geq 1$. If $\lim_{x\to\infty} f(x) = A$, then $\lim_{n\to\infty} a_n = A$.
- Example 5.

$$\lim_{n\to\infty} \frac{n+1}{n} = \lim_{x\to\infty} \frac{x+1}{x} = 1.$$

$$\lim_{n\to\infty} \frac{\sqrt{n^2-1}}{n} = \lim_{x\to\infty} \frac{\sqrt{x^2-1}}{x} = 1.$$

One uses L'Hopital's rule to derive the limit of the above two functions.

Sequences

- $$\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$
  $$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$$
  $$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$
  $$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$
  if $\lim_{n\to\infty} b_n \neq 0$.

Sequences

- Property c: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.
  - Example 5.

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0.$$

Sequences

- Property d (Squeezing Theorem): If $0 \leq a_n \leq b_n$, $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$.
  - Example 6. Determine if

$$\lim_{n\to\infty} \frac{\sin(\sqrt{n^2-1})}{n}$$

converges or diverges.

Solution: It converges because

$$|\frac{\sin(\sqrt{n^2-1})}{n}| \leq \frac{1}{n}$$

and $\lim_{n\to\infty} \frac{1}{n} = 0$

Sequences

- Fact d' (Squeezing Theorem): If $c_n \leq a_n \leq b_n$, $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$, then $\lim_{n\to\infty} a_n = 0$.
  - Example 7. Determine if

$$\lim_{n\to\infty} a_n$$

converges or diverges, where $a_n = \frac{(-1)^n}{n}$ if n is odd; and $a_n = \frac{(-1)^n}{n^3}$ if n is even.

Sequences

- Solution: Note

$$-\frac{1}{n} \leq a_n \leq \frac{1}{n^3},$$

$$\lim_{n\to\infty} -\frac{1}{n} = 0$$

and

$$\lim_{n\to\infty} \frac{1}{n^3} = 0.$$

By Squeezing Theorem, the sequence converges.