Series

Definition

\( a_1 + a_2 + a_3 + \cdots \) is called an infinite series or just series.

Denoted by

\[
\sum_{n=1}^{\infty} a_n, \text{ or } \sum a_n.
\]
Given a series $\sum_{n=1}^{\infty} a_n$, let $s_n$ denote its partial sum 

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$ 

If $\lim_{n \to \infty} s_n$ exists as a finite number, then the series 

$$\sum_{n=1}^{\infty} a_n := \lim_{n \to \infty} s_n,$$ 

and we say it is convergent.

If $\lim_{n \to \infty} s_n$ does not exist, we say $\sum_{n=1}^{\infty} a_n$ is divergent.
Example 1. Suppose \( s_n = \frac{3^n}{2n+3} \). Then

\[
a_n = s_n - s_{n-1} = \frac{3n}{2n+3} - \frac{3(n-1)}{2(n-1)+3} = \frac{3n}{2n+3} - \frac{3n-3}{2n+1}.
\]

\[
\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{3n}{2n+3} = \frac{3}{2}.
\]
Example 2. Geometric series $a_n = a \cdot r^{n-1}$. Compute $\sum_{n=1}^{\infty} a_n$ for $-1 < r < 1$.

Remark: in other words, $\{a_n\}$ is a geometric series if $\frac{a_{n+1}}{a_n}$ equals $r$ for all $n$.

Solution:

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1} = a \frac{1 - r^n}{1 - r}, \quad (1)$$

for $r \neq 1$. 

How to prove identity (1)?

\[ s_n = a + ar + ar^2 + \cdots + ar^{n-1} \]

\[ r \cdot s_n = ar + ar^2 + ar^3 + \cdots + ar^n \]

Thus

\[ (1 - r)s_n = a - ar^n \]

which is equivalent to

\[ s_n = a \frac{1 - r^n}{1 - r}. \]
If \(-1 < r < 1\), then \(r^n \to 0\). Thus

\[
\lim_{n \to \infty} s_n = a \frac{1}{1 - r}.
\]
If \( r = -1 \), then limit of \( s_n = a \frac{1-r^n}{1-r} \) does not exist.
If \( r = 1 \), then the partial sum \( s_n \) is not equal to \( \frac{1}{1-r} \). It should be \( s_n = n \) whose limit is infinity.
If \( r \leq -1 \) or \( r > 1 \), then limit of \( r^n \) does not exist. Hence limit of \( s_n \) does not exist.
Conclusion: If $-1 < r < 1$, then $\sum_{n=1}^{\infty} ar^{n-1} = a\frac{1}{1-r}$. The series converges.

If $r \leq -1$ or $r \geq 1$, then $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.
Example 3. Find the sum of geometric series

\[ 5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots. \]

Solution: Note \( \frac{a_{n+1}}{a_n} \) is a constant for all \( n \), and they all equal to \(-\frac{2}{3}\). Thus it is a geometric series. \( a = 5, \ r = \frac{a_{n+1}}{a_n} = \frac{a_2}{a_1} = -\frac{2}{3} \).

Apparently, \(|r| < 1\).

Thus the series equals to

\[ a \frac{1}{1 - r} = 5 \cdot \frac{1}{1 - \left(-\frac{2}{3}\right)} = \frac{5}{1 + \frac{2}{3}} = 3. \]
Example 4. Is the series $\sum_{n=1}^{\infty} 5^{2n}2^{1-n}$ convergent or divergent?

Solution:

$$\sum_{n=1}^{\infty} 5^{2n}2^{1-n} = \sum_{n=1}^{\infty} (5^2)^n \cdot 2 \cdot \frac{1}{2^n} = \sum_{n=1}^{\infty} 2 \cdot \left(\frac{25}{2}\right)^n.$$ 

Note $r = \frac{25}{2} > 1$, thus the geometric series diverges.
Example 5. Find \( \sum_{n=2}^{\infty} 2^{-n} \).

Solution:

\[
\sum_{n=2}^{\infty} 2^{-n} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots = \frac{1}{4}(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots) = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}.
\]

Thus

\[
\sum_{n=2}^{\infty} 2^{-n} = \frac{1}{4} \cdot \frac{1}{1 - \frac{1}{2}} = \frac{1}{2}.
\]
Example 6. Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent. And it is equal to 1.

Solution:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} \cdots = 1.$$ 

Very few series have such a beautiful cancellation formula. For most of the series, we cannot compute the partial sum explicitly. See the next example.