Improper Integrals: Part 2

Sometimes it is difficult to find the exact value of an improper integral, but we can still know if it is convergent or divergent by comparing it with some other improper integral.

- Comparison test:

  Suppose f and g are continuous with $f(x) \geq g(x) \geq 0$, for $x \geq a$.

  - (1). If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.
  - (2)(Equivalent statement). If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

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- Comparison test:

  (3). If $\int_a^\infty |f(x)|dx$ is convergent, then $\int_a^\infty f(x)dx$ is convergent.

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- Example 5. Determine if the integral

$$\int_0^{2\pi} \frac{\cos x}{\sqrt{x}} dx.$$

is convergent or divergent.

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- Solution: Compare $\frac{|\cos x|}{\sqrt{x}}$ with $\frac{1}{\sqrt{x}}$ on $[0, 2\pi]$.

$$\frac{|\cos x|}{\sqrt{x}} \leq \frac{1}{\sqrt{x}},$$

and $\int_0^{2\pi} \frac{1}{\sqrt{x}} dx$ converges, thus by comparison test (1) $\int_0^{2\pi} \frac{|\cos x|}{\sqrt{x}} dx$ converges.

Next, by comparison test (3) $\int_0^{2\pi} \frac{\cos x}{\sqrt{x}} dx$ converges.

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- Example 6. Determine if the integral

$$\int_0^\infty \frac{1}{x^5 + 3x^2 + 2x + 1}\,dx.$$

is convergent or divergent.

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- Solution:
$$\frac{1}{x^5+3x^2+2x+1} \le \frac{1}{x^5}$$
and $\int_1^\infty \frac{1}{x^5}dx$ converges because $p = 5 > 1$. Thus by comparison test (1), $\int_1^\infty \frac{1}{x^5+3x^2+2x+1}dx$ converges.
Now,
$$\begin{aligned}&\int_0^\infty \frac{1}{x^5+3x^2+2x+1}dx \\ =&\int_0^1 \frac{1}{x^5+3x^2+2x+1}dx + \int_1^\infty \frac{1}{x^5+3x^2+2x+1}dx.\end{aligned} \tag{2}$$
Thus it is convergent. (I split the integral into two parts, because 0 is a discontinuous point of $\frac{1}{x^5}$, and thus I want to avoid this point when applying comparison test.)

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- Example 7. Determine if the integral

$$\int_{10}^{\infty} \frac{1}{x^5 - 3x^2 + 2x + 1} dx.$$

is convergent or divergent.

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- Idea:

$$\frac{1}{x^5 - 3x^2 + 2x + 1} \leq \frac{1}{x^5}$$

does not hold any more. However,

$$\frac{1}{x^5 - 3x^2 + 2x + 1} \leq \frac{1}{0.9x^5}$$

when $x > 0$ is very big. This can be seen by comparing the coefficient of the leading order term x^5 in the two denominators.

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- Solution:
$$\frac{1}{x^5 - 3x^2 + 2x + 1} \leq \frac{1}{0.9x^5}$$
when $x > 0$ is very big, and $\int_{10}^{\infty} \frac{1}{0.9x^5} dx$ converges because $p = 5 > 1$. Thus by comparison test (1), $\int_{10}^{\infty} \frac{1}{x^5 - 3x^2 + 2x + 1} dx$ converges.

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- Example 8. Determine if the integral

$$\int_0^{\infty} e^{-x^2} dx.$$

is convergent or divergent.

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- Idea: It is hard to find antiderivative of e^{-x^2}. But we can try comparison test to determine if it is convergent without computing the value of the integral.

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- Solution:

$$e^{-x^2} \leq e^{-x}$$

for $x \geq 1$. Also $\int_1^\infty e^{-x}dx$ converges (by direct computation). Thus $\int_1^\infty e^{-x^2}dx$ converges by comparison test (1).
Now

$$\int_0^\infty e^{-x}dx = \int_0^1 e^{-x}dx + \int_1^\infty e^{-x}dx.$$

The first term is finite as it is a (regular) integral of a continuous function on $[0,1]$. The second term is convergent due to the previous discussion using comparison test.