Improper Integrals: Part 2

The second type of improper integral: the interval is finite, but the integrand is discontinuous at some points.

- If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx := \lim_{t \to b-} \int_a^t f(x)dx.$$

If the limit exists as a finite number, we say this improper integral converges, otherwise we say it diverges.

Improper Integrals: Part 2

- If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx := \lim_{t \to a+} \int_t^b f(x)dx.$$

If the limit exists as a finite number, we say this improper integral converges, otherwise we say it diverges.

Improper Integrals: Part 2

- Example 1. Find the area under the curve $y = \frac{1}{x}$ for $0 < x \leq 1$.
  - Remark: This is not an ordinary definite integral, since $\frac{1}{x}$ is not well-defined at 0. It goes to ∞. See the graph.

Improper Integrals: Part 2

Solution: By definition,

$$\int_0^1 \frac{1}{x} dx := \lim_{t \to 0+} \int_t^1 \frac{1}{x} dx = \lim_{t \to 0+} \ln x|_t^1 = 0 - (-\infty) = \infty.$$

Thus it diverges.

- We call the improper integral is divergent if the limit does not exist.

Improper Integrals: Part 2

- Example 2.(Important!) Find the value of p so that $\int_0^1 \frac{1}{x^p} dx$ converges.

  Solution:

$$\int_0^1 \frac{1}{x^p} dx := \lim_{t \to 0+} \int_t^1 \frac{1}{x^p} dx = \lim_{t \to 0+} \frac{x^{1-p}}{1-p} \Big|_t^1 = \frac{1}{1-p} - \lim_{t \to 0+} \frac{t^{1-p}}{1-p}.$$

  Two cases:

  i) if $p < 1$, then $\lim_{t \to 0+} \frac{t^{1-p}}{1-p} = 0$. This improper integral is convergent.

  ii) if $p > 1$, then $\lim_{t \to 0+} \frac{t^{1-p}}{1-p} = \infty$. This improper integral is divergent.

Improper Integrals: Part 2

Therefore $\int_0^1 \frac{1}{x^p} dx$ converges when $p < 1$, $\int_0^1 \frac{1}{x^p} dx$ diverges when $p \geq 1$. ($p = 1$ case is discussed in Example 1.)

Improper Integrals: Part 2

- Example 3. Evaluate

$$\int_0^3 \frac{dx}{x-1}.$$

Note $\frac{1}{x-1}$ is discontinuous at $x = 1$ and $x = 1$ not one of the end points of $[1, 3]$.

- When discontinuity occurs at an interior point c of the interval $[a, b]$, we have to split the integral into $[a, c]$ and $[c, b]$ so that discontinuous point is at one of the end points of the interval. This way, we can use the definition of the improper integral.

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1} = \lim_{t \to 1^-} \int_0^t \frac{dx}{x-1} + \lim_{t \to 1^+} \int_t^3 \frac{dx}{x-1}.$$

Improper Integrals: Part 2

$$\lim_{t\to 1^-}\int_0^t \frac{dx}{x-1} = \lim_{t\to 1^-} \ln|t-1| - \ln 1 = -\infty.$$

$$\lim_{t\to 1^+}\int_t^3 \frac{dx}{x-1} = \ln|2| - \lim_{t\to 1^+} \ln|t-1| = \infty.$$

Thus the improper integral is divergent.

- Rule: when breaking an improper integral into two (or more) integrals by splitting the interval, if one of the improper integral diverges, then this improper integral diverges.
- Note it is incorrect that $(\infty) + (-\infty) = 0$.

Improper Integrals: Part 2

- Example 4 Evaluate

$$\int_0^1 \ln x dx.$$

Solution:

$$\begin{aligned} \int_0^1 \ln x dx &= \lim_{t\to 0^+} \int_t^1 \ln x dx \\ &= \lim_{t\to 0^+} [x \ln x - x]|_t^1 \qquad (1) \\ &= \lim_{t\to 0^+} -1 - t\ln t + t. \end{aligned}$$

Improper Integrals: Part 2

By L'Hospital's rule,

$$\lim_{t \to 0^+} t \ln t = 0.$$

Thus the integral equals -1

Improper Integrals: Part 2

Sometimes it is difficult to find the exact value of an improper integral, but we can still know if it is convergent or divergent by comparing it with some other improper integral.

- Comparison test:

  Suppose f and g are continuous with $f(x) \geq g(x) \geq 0$, for $x \geq a$.

  - (1). If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.
  - (2)(Equivalent statement). If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Improper Integrals: Part 2

- Example 4. Determine if the integral

$$\int_1^{\infty} \frac{1+\sin^2 x}{x} dx.$$

is convergent or divergent.

Improper Integrals: Part 2

- Solution: Compare $\frac{1+\sin^2 x}{x}$ with $\frac{1}{x}$ on $[1, \infty)$.

$$\frac{1+\sin^2 x}{x} \geq \frac{1}{x},$$

and $\int_1^{\infty} \frac{1}{x} dx$ diverges, thus $\int_1^{\infty} \frac{1+\sin^2 x}{x} dx$ also diverges using (2).