Improper Integrals: Part 2

Sometimes it is difficult to find the exact value of an improper integral, but we can still know if it is convergent or divergent by comparing it with some other improper integral.

Comparison test:

Suppose \( f \) and \( g \) are continuous with \( f(x) \geq g(x) \geq 0 \), for \( x \geq a \).

\( (1) \). If \( \int_a^\infty f(x)\,dx \) is convergent, then \( \int_a^\infty g(x)\,dx \) is convergent.

\( (2) \). If \( \int_a^\infty g(x)\,dx \) is divergent, then \( \int_a^\infty f(x)\,dx \) is divergent.
Comparison test:

(3). If $\int_a^\infty |f(x)| \, dx$ is convergent, then $\int_a^\infty f(x) \, dx$ is convergent.
Example 4. Determine if the integral

\[ \int_{1}^{\infty} \frac{1 + \sin^2 x}{x} \, dx. \]

is convergent or divergent.

Solution: Compare \( \frac{1+\sin^2 x}{x} \) with \( \frac{1}{x} \) on \([1, \infty)\).

\[ \frac{1 + \sin^2 x}{x} \geq \frac{1}{x}, \]

and \( \int_{1}^{\infty} \frac{1}{x} \, dx \) diverges, thus \( \int_{1}^{\infty} \frac{1+\sin^2 x}{x} \, dx \) also diverges using (2).
Example 5. Determine if the integral

$$\int_{0}^{1} \frac{\cos x}{\sqrt{x}} \, dx.$$  

is convergent or divergent.

Solution: Compare $\frac{|\cos x|}{\sqrt{x}}$ with $\frac{1}{\sqrt{x}}$ on $[0, 1]$.

$$\frac{|\cos x|}{\sqrt{x}} \leq \frac{1}{\sqrt{x}},$$

and $\int_{0}^{1} \frac{1}{\sqrt{x}} \, dx$ converges, thus $\int_{0}^{1} \frac{|\cos x|}{\sqrt{x}} \, dx$ also converges using (1).

By (3), $\int_{0}^{1} \frac{\cos x}{\sqrt{x}} \, dx$ converges.
Chapter 11: Sequences and Series

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Definition: A sequence (of numbers) is a list of \( \{a_1, a_2, a_3, \cdots \} \)
linearly ordered by an index set \( I \). \( I \) is just the set of positive integers.

Other ways to write a sequence:

\[ \{a_n\}_{n=1}^{\infty}, \]

or

\[ \{a_n\}_{n\in \mathbb{N}_+}. \]
The numbers are not necessarily distinct.

Example 1. $a_1 = 1$, $a_2 = 0$, $a_3 = 0$, $\cdots$ is a sequence.
For some sequences, there is a function $f(x) : \mathbb{Z}^+ \rightarrow \mathbb{R}$, such that $a_n = f(n)$.

Example 2. $a_1 = 1, a_2 = 3, a_3 = 5, a_4 = 7 \cdots$ is a sequence.

In this example, $f(x) = 2x - 1$. Another way to write it is

$$\{2n - 1\}_{n=1}^{\infty}.$$
Example 3. \( a_1 = 2, a_2 = 4, a_3 = 6, a_4 = 8 \cdots \) is a sequence.

In this example, \( f(x) = 2x \). Another way to write it is

\[
\{2n\}_{n=1}^\infty.
\]
Some more examples of sequences.

\[ \{n^2\}^{\infty}_{n=1}, \]
\[ \{\sqrt{n+1}\}^{\infty}_{n=1}, \]
\[ \{\cos\left(\frac{(n+1)\pi}{2}\right)\}^{\infty}_{n=1}, \]
\[ \left\{\frac{1}{n} - \frac{1}{n+1}\right\}^{\infty}_{n=1}. \]
Sequences

There are also examples that we cannot write down the general formula of the $n$-th term in the sequence. Example 4.

$$a_1 = 1, \ a_2 = 2,$$

$$a_n = a_{n-1} + a_{n+1} \text{ for } n \geq 3.$$

This is the famous Fibonacci sequence.
Sequences

There are 3 ways to express a sequence:

1. list all the terms
2. a function $f : \mathbb{Z}^+ \rightarrow \mathbb{R}$ such that $a_n = f(n)$
3. an inductive formula.
We will put a lot of effort to understand asymptotic behavior of a sequence, namely as $n \to \infty$.

Suppose $f(x)$ is a function so that $a_n = f(n)$ for all $n \geq 1$. If $\lim_{x \to \infty} f(x) = A$, then $\lim_{n \to \infty} a_n = A$.

Example 5.

\[
\lim_{n \to \infty} \frac{n + 1}{n} = 1.
\]

\[
\lim_{n \to \infty} \frac{\sqrt{n^2 - 1}}{n} = 1.
\]

\[
\lim_{n \to \infty} \frac{\sin \sqrt{n^2 - 1}}{n} = 0.
\]
Sequences

\[
\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n
\]

\[
\lim_{n \to \infty} c a_n = c \lim_{n \to \infty} a_n
\]

\[
\lim_{n \to \infty} (a_n b_n) = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n
\]

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n}
\]

if \(\lim_{n \to \infty} b_n \neq 0\).