AS.110.109: Calculus II (Eng)

Chapter 7.8: Improper Integral

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Improper Integrals

- We use to study bounded function $f(x)$'s definite integrals over a finite interval $[a, b]$. When There are two types of improper integrals.
  - ▶ The interval is infinite . (Today's lecture)
  - ▶ The function is discontinuous at some points. (Next Monday's lecture)

Improper Integrals: Part 1

- Definition of improper integral:

$$\int_a^\infty f(x)dx := \lim_{b\to\infty} \int_a^b f(x)dx$$

  if the limit exists ("exists" means "limit exists as a finite number").
- The improper integral $\int_a^\infty f(x)dx$ is convergent if the limit exists.
  It is divergent if the limit does not exist.

Improper Integrals: Part 1

$\int_{-\infty}^{b} f(x)dx$ is defined similarly.

$$\int_{-\infty}^{b} f(x)dx := \lim_{a \to -\infty} \int_{a}^{b} f(x)dx.$$

- Unlike the definite integral, we do not plug in ∞ to evaluate directly. Instead, we take limit of definite integrals on larger and larger intervals, approximating $[a, \infty)$.

Improper Integrals: Part 1

- Example 1. Compute the improper integral $\int_a^\infty c dx$, for some constant $c \neq 0$. Does it converge or diverge?

  Solution:

$$\int_a^b f(x)dx = c(b-a).$$

By definition of improper integral,

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx = \lim_{b\to\infty} c(b-a) = \infty.$$

The area under the curve $f(x) = c$ is infinite.

Improper Integrals: Part 1

- Geometric meaning:
  $\int_a^b f(x)dx$ is the area under the graph of $f(x)$ from a to b.
  $\int_a^\infty f(x)dx$ is the area under the graph of $f(x)$ from a to ∞.

Improper Integrals: Part 1

Thus in order to have a finite area under the curve, the curve should decrease to 0 in some sense at ∞.

- Example 2. Compute the improper integral $\int_a^\infty \frac{1}{x} dx$. Does it converge or diverge?

  Solution: By definition,

$$\int_a^\infty \frac{1}{x} dx := \lim_{b\to\infty} \int_a^b \frac{1}{x} dx = \lim_{b\to\infty} \ln b - \ln a = \infty.$$

The improper integral is divergent.

Improper Integrals: Part 1

So even though $\frac{1}{x}$ is decreasing to 0 at ∞, it still does not guarantee the area under its graph if we integrate over $[a, \infty)$ is a finite number.

Improper Integrals: Part 1

- Example 3. Find the values of p so that the improper integral $\int_1^\infty \frac{1}{x^p} dx$ converges.

  Solution: By definition,

$$\int_1^\infty \frac{1}{x^p} dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^p} dx$$

Improper Integrals: Part 1

If $p \neq 1$, then it equals

$$\lim_{b\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^b = \lim_{b\to\infty} \frac{b^{1-p}-1}{1-p}.$$

This last limit converges to $\frac{1}{p-1}$ if $p > 1$.

The limit diverges (equals ∞) if $p < 1$.

Improper Integrals: Part 1

What happens at $p = 1$?

$$\int_1^\infty \frac{1}{x} dx = \lim_{b \to \infty} \int_1^b \frac{1}{x} dx = \lim_{b \to \infty} \ln x \Big|_1^b = \infty.$$

Thus at $p = 1$, the improper integral also diverges.

- Conclusion: $\int_1^\infty \frac{1}{x^p} dx$ converges if $p > 1$;
  $\int_1^\infty \frac{1}{x^p} dx$ diverges if $p \leq 1$.

Improper Integrals: Part 1

- Example 4. Compute the improper integral $\int_1^\infty \frac{\ln x}{x^2} dx$. Does it converge or diverge?

  Solution: Use integration by parts,

$$\int_1^\infty \frac{\ln x}{x^2} dx = \lim_{b\to\infty} \ln x \cdot \frac{-1}{x}|_1^b + \int_1^b \frac{1}{x} \cdot \frac{1}{x} dx = \lim_{b\to\infty} [\frac{-\ln x}{x} - \frac{1}{x}]|_1^b.$$

Improper Integrals: Part 1

By the L'Hospital's rule,

$$\lim_{b\to\infty} \frac{-\ln b}{b} = \lim_{b\to\infty} \frac{-1}{b} = 0$$

Thus the improper integral equals 1. It converges.