Areas and lengths in polar coordinates

- Area between two polar curves $r = f(\theta)$ and $r = g(\theta)$ for $\theta \in [\theta_1, \theta_2]$ is

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} f^2(\theta) - \frac{1}{2} g^2(\theta) d\theta.$$

■ Example 2. Given a polar curve $r = 2\sin\theta$ and $r = 1 + \sin\theta$ for $\theta \in [\frac{\pi}{4}, \frac{3\pi}{4}]$. Compute the area of the polar region.

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Solution: Since $1+\sin\theta \geq 2\sin\theta$,

$$A = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{2}(1+\sin\theta)^2 - \frac{1}{2}(2\sin\theta)^2 d\theta \qquad (8)$$

$$= \cdots .$$

Areas and lengths in polar coordinates

Arc length in parametric curve

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2} dt.$$

Areas and lengths in polar coordinates

Polar curve $r = f(\theta)$ for $\theta \in [a, b]$ gives parametric equations:

$$x = r\cos\theta = f(\theta)\cos\theta;$$

$$y = r\sin\theta = f(\theta)\sin\theta$$

with $\theta \in [a, b]$.

$$L = \int_a^b \sqrt{(x'(\theta))^2 + (y'(\theta))^2}\,d\theta.$$

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By chain rule,

$$x'(\theta) = f' \cos\theta - f \sin\theta;$$

$$y'(\theta) = f' \sin\theta + f \cos\theta.$$

Thus

$$\begin{aligned} L &= \int_a^b \sqrt{(f' \cos\theta - f \sin\theta)^2 + (f' \sin\theta + f \cos\theta)^2} d\theta \\ &= \int_a^b \sqrt{(f'(\theta))^2 + f^2(\theta)} d\theta. \end{aligned} \tag{9}$$

Areas and lengths in polar coordinates

Example 3. Find the length of the cardioid $r = 1 + \sin\theta$.

Solution:

$$
\begin{aligned}
L &= \int_0^{2\pi} \sqrt{(f'(\theta))^2 + f^2(\theta)}d\theta \\
&= \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}d\theta \\
&= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}d\theta \\
&= \ldots
\end{aligned}
\tag{10}
$$