Chapter 7.8: Improper Integral

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Fall 2017
We use to study bounded function $f(x)$’s definite integrals over a finite interval $[a, b]$. When there are two types of improper integrals.

- The interval is infinite. (Today’s lecture)
- The function is discontinuous at some points. (Next Monday’s lecture)
Definition of improper integral:

\[ \int_{a}^{\infty} f(x) \, dx := \lim_{b \to \infty} \int_{a}^{b} f(x) \, dx \]

if the limit exists (“exists” means “limit exists as a finite number”).

The improper integral \( \int_{a}^{\infty} f(x) \, dx \) is convergent if the limit exists. It is divergent if the limit does not exist.
Improper Integrals: Part 1

\[ \int_{-\infty}^{b} f(x) \, dx \] is defined similarly.

\[ \int_{-\infty}^{b} f(x) \, dx := \lim_{a \to -\infty} \int_{a}^{b} f(x) \, dx. \]

Unlike the definite integral, we do not plug in \( \infty \) to evaluate directly. Instead, we take limit of definite integrals on larger and larger intervals, approximating \([a, \infty)\).
Example 1. Compute the improper integral $\int_{a}^{\infty} c \, dx$, for some constant $c \neq 0$. Does it converge or diverge?

Solution:

\[ \int_{a}^{b} f(x) \, dx = c(b - a). \]

By definition of improper integral,

\[ \int_{a}^{\infty} f(x) \, dx = \lim_{b \to \infty} \int_{a}^{b} f(x) \, dx = \lim_{b \to \infty} c(b - a) = \infty. \]

The area under the curve $f(x) = c$ is infinite.
Geometric meaning:

\[ \int_{a}^{b} f(x)\,dx \] is the area under the graph of \( f(x) \) from \( a \) to \( b \).

\[ \int_{a}^{\infty} f(x)\,dx \] is the area under the graph of \( f(x) \) from \( a \) to \( \infty \).
Thus in order to have a finite area under the curve, the curve should decrease to 0 in some sense at $\infty$.

Example 2. Compute the improper integral $\int_{a}^{\infty} \frac{1}{x} \, dx$. Does it converge or diverge?

Solution: For any finite interval $[a, b]$, $a > 0$

$$\int_{a}^{b} \frac{1}{x} \, dx = \ln b - \ln a.$$
By definition,

\[ \int_{a}^{\infty} \frac{1}{x} \, dx := \lim_{b \to \infty} \int_{a}^{b} \frac{1}{x} \, dx = \infty. \]

The improper integral is divergent.
Improper Integrals: Part 1

So even $\frac{1}{x}$ is decreasing to 0 at $\infty$, it still does not guarantee the area under its graph if we integrate over $[a, \infty)$ is a finite number.
Example 3. Find the values of $p$ so that the improper integral

$$\int_1^\infty \frac{1}{x^p} \, dx$$

converges.

Solution: By definition,

$$\int_1^\infty \frac{1}{x^p} \, dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^p} \, dx$$
If $p \neq 1$, then it equals

$$\lim_{b \to \infty} \frac{x^{1-p}}{1 - p}\bigg|_1^b = \lim_{b \to \infty} \frac{b^{1-p} - 1}{1 - p}.$$ 

This last limit converges to $\frac{1}{p-1}$ if $p > 1$. The limit diverges (equals $\infty$) if $p < 1$. 
What happens at $p = 1$?

$$\int_1^\infty \frac{1}{x} \, dx = \lim_{b \to \infty} \int_1^b \frac{1}{x} \, dx = \lim_{b \to \infty} \ln x \bigg|_1^b = \infty.$$ 

Thus at $p = 1$, the improper integral also diverges.

**Conclusion:** $\int_1^\infty \frac{1}{x^p} \, dx$ converges if $p > 1$;

$\int_1^\infty \frac{1}{x^p} \, dx$ diverges if $p \leq 1$. 

Example 4. Compute the improper integral \( \int_{1}^{\infty} \frac{\ln x}{x^2} \, dx \). Does it converge or diverge?

Solution: Use integration by parts,

\[
\int_{1}^{\infty} \frac{\ln x}{x^2} \, dx = \ln x \cdot \frac{-1}{x} \bigg|_{1}^{\infty} + \int_{1}^{\infty} \frac{1}{x} \cdot \frac{1}{x} \, dx = \left[ \frac{-\ln x}{x} - \frac{1}{x} \right]_{1}^{\infty}.
\]
By the L’Hospital’s rule,

\[ \lim_{x \to \infty} \frac{-\ln x}{x} = \lim_{x \to \infty} \frac{-1}{x} = 0 \]

Thus the improper integral equals 1. It converges.