Polar Coordinates

Polar Coordinates:

Given any point $P = (x, y)$ on the plane

- r stands for the distance from the origin $(0, 0)$.
- θ stands for the angle from positive x-axis to OP.

Polar coordinate: (r, θ)

Polar Coordinates

- One can easily convert Cartesian coordinate (x, y) to polar coordinates (r, θ), and the other way around.

  r is a function of x, y;

  θ is also a function of x, y.

$$r = \sqrt{x^2 + y^2};$$

$$\theta = \tan^{-1} \frac{y}{x}.$$

Polar Coordinates

x is a function of r and θ;

y is a function of r and θ.

$$x = r\cos\theta;$$

$$y = r\sin\theta.$$

Polar Coordinates

- Example 1. Convert the point with polar coordinates $(2, \pi)$ to Cartesian coordinates.

  Solution: Since $r = 2$, $\theta = \pi$, in Cartesian coordiates

$$x = r\cos\theta = 2\cos\pi = -2;$$

$$y = r\sin\theta = 2\sin\pi = 0.$$

  Thus the point in Cartesian coordinates is $(-2, 0)$.

Polar Coordinates

- Example 2. Convert the point with Cartesian coordinates $(-2, 2\sqrt{3})$ to polar coordinates.
  Solution: Since $x = -2$, $y = 2\sqrt{3}$, in Cartesian coordinates

$$r = \sqrt{x^2 + y^2} = \sqrt{4 + 4 \cdot 3} = \sqrt{16} = 4;$$

$$\theta = \tan^{-1} \frac{y}{x} = \tan^{-1}(-\sqrt{3}) = 2\pi/3.$$

  Thus the point in Cartesian coordinates is $(4, 2\pi/3)$.

Polar Coordinates

- Polar curve: the graph of a polar equation $r = f(\theta)$, or $F(r, \theta)$ consist of all points P that satisfies the equation.

Polar Coordinates

- Example 3. Convert the polar equation $r = 3$ to a Cartesian equation, and sketch the curve.

  Solution: Since $r = \sqrt{x^2 + y^2}$, the Cartesian equation is

  $$x^2 + y^2 = 9.$$

  This is a circle of radius 3. Easy to sketch.

Polar Coordinates

- Example 4. Convert the polar equation $r = 3\sin\theta$ to a Cartesian equation, and sketch the curve.

  Solution: $r = 3\sin\theta$ and $y = r\sin\theta$ imply that $r = 3\frac{y}{r}$. Thus

$$r^2 = 3y.$$

  To eliminate r, we write $r = \sqrt{x^2 + y^2}$. Hence

$$x^2 + y^2 = 3y.$$

  This is a Cartesian equation.

Polar Coordinates

What is this curve? Complete the square

$$x^2 + (y - \frac{3}{2})^2 = (\frac{3}{2})^2.$$

Thus the curve is a circle, centered at $(0, \frac{3}{2})$ with radius $\frac{3}{2}$.

Areas and lengths in polar coordinates

Area formula revisited:

- Area between the graph of $f(x)$ and x-axis is given by

$$A = \int_a^b |f(x)| dx.$$

Areas and lengths in polar coordinates

- Area formula of the region between the graph of $f(x)$ and $g(x)$ is given by

$$A = \int_a^b |f(x) - g(x)| dx.$$

Areas and lengths in polar coordinates

Let $r = f(\theta)$ for $\theta \in [a, b]$ be a curve in the plane. The polar region is the region enclosed by the ray $\theta = \theta_1$, $\theta = \theta_2$ and the curve $r = f(\theta)$ for $\theta \in [\theta_1, \theta_2]$.

How to compute the area?

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 d\theta = \int_{\theta_1}^{\theta_2} \frac{1}{2} f^2(\theta) d\theta.$$

Areas and lengths in polar coordinates

- Example 1. Given a polar curve $r = 3\sin\theta$ for $\theta \in [\frac{\pi}{4}, \frac{3\pi}{4}]$.

  Compute the area of the polar region.

  Solution: By Example 4 in Chapter 10.3, polar curve $r = 3\sin\theta$ is a circle.

$$\begin{aligned} A =& \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{2}(3\sin\theta)^2 d\theta \\ =& \frac{9}{4}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 1 - \cos 2\theta d\theta \\ =& \frac{9}{4}(\frac{\pi}{2} - \frac{1}{2}\sin 2\theta|_{\frac{\pi}{4}}^{\frac{3\pi}{4}}) \\ =& \frac{9}{4}(\frac{\pi}{2} + 1). \end{aligned} \tag{7}$$