The integral test and estimates of sums

- Example 4. Determine if the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.
  - Solution: Use the integral test. $f(x) = \frac{\ln x}{x}$ is positive

continuous for $x > 1$. It is not obvious if f is decreasing or not. Thus we compute f'.

$$f'(x) = \frac{1 - \ln x}{x^2},$$

thus $f'(x) < 0$ when $\ln x > 1$, that is $x > e$.

The integral test and estimates of sums

So we can apply the integral test (on $[e, \infty)$):

$$\int_e^\infty \frac{\ln x}{x} dx = \lim_{t\to\infty} \int_e^t \frac{\ln x}{x} dx = \lim_{t\to\infty} \frac{(\ln x)^2}{2}|_e^t = \infty.$$

We know $e = 2.71828182846$ Thus $\sum_{n=3}^{\infty} \frac{\ln n}{n}$ diverges. Of course, this implies $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges.

The integral test and estimates of sums

- Remark: We learn from this example that to determine the convergence or divergence, we don't mind the first finite number of terms in the series. Thus in the integral test, it is only necessary to assume $f(x)$ is decreasing near infinity.
  Namely, there exists a constant A, such that

$$f'(x) < 0 \qquad \text{for} \quad x > A.$$

The integral test and estimates of sums

Example 5. Determine if the series $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ converges or diverges.

Solution: Use the integral test.

Use $f(x) = x^2 e^{-x^3}$. f is positive continuous. Is it decreasing?

$$f'(x) = e^{-x^3} x(2 - 3x^3) < 0$$

if $x \geq 1$.

The integral test and estimates of sums

Substitute $u = x^3$. Then

$$\int_1^\infty x^2 e^{-x^3} dx = \frac{1}{3}\int_1^\infty e^{-u} du = -\frac{1}{3}e^{-u}|_1^\infty = \frac{1}{3}e^{-1} < \infty.$$

Thus the series $\sum_{n=1}^\infty n^2 e^{-n^3}$ is convergent.

The comparison test

- The idea of the comparison test is to compare a given series with a series that is known to be convergent or divergent.

  Example 1. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{3^n + 2}$ is convergent or divergent.

  Solution: The series $\sum_{n=1}^{\infty} \frac{1}{3^n + 2}$ reminds us of the series $\sum_{n=1}^{\infty} \frac{1}{3^n}$. The latter is a geometric series with $a = \frac{1}{3}$, $r = \frac{1}{3}$, and thus it is convergent.

The comparison test

We can compare each term in these two series:

$$\frac{1}{3^n + 2} < \frac{1}{3^n}.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{3^n + 2} < \sum_{n=1}^{\infty} \frac{1}{3^n}.$$

So it is convergent.

The comparison test

The comparison test Suppose $0 \leq a_n \leq b_n$

(a) If $\sum b_n$ is convergent, then $\sum a_n$ is also convergent.

(b) If $\sum a_n$ is divergent, then $\sum b_n$ is also divergent.

The comparison test

We must have some known series $\sum b_n$ for the purpose of comparison. Usually, we compare with one of the following two series:

(a) Geometric series $\sum_{n=1}^{\infty} a \cdot r^n$ converges for $|r| < 1$ and diverges for all other values of r;

(b) A p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for all other values of p;

The comparison test

- Example 2. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^2+n+4}$ is convergent or divergent.
  - Solution: The leading order term as n tends to ∞ is $\frac{1}{n^2}$.

$$\frac{1}{n^2+n+4} \leq \frac{1}{n^2}.$$

Note the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (because $p = 2 > 1$).

Therefore by comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^2+n+4}$ converges.

The comparison test

- Example 3. Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n + 1}{\sqrt{n}}$ is convergent or divergent.

  Solution: Compare with $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n + 1}{\sqrt{n}}$. Compare with $\frac{1}{\sqrt{n}}$:

$$\frac{\ln n + 1}{\sqrt{n}} \geq \frac{1}{\sqrt{n}}.$$

  Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, by comparison test $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n + 1}{\sqrt{n}}$ also diverges.

The comparison test

- Example 4. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^2 - n}$ is convergent or divergent.
  - Solution: The leading order term as n tends to ∞ is $\frac{1}{n^2}$. Note the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (because $p = 2 > 1$). So if we could prove

$$\frac{1}{n^2 - n} \leq \frac{1}{n^2}, \tag{4}$$

then the series converges. But this inequality (3) is not true.

The comparison test

In fact,

$$\frac{1}{n^2} \leq \frac{1}{n^2 - n}. \tag{5}$$

But

$$\lim_{n\to\infty} \frac{\frac{1}{n^2-n}}{\frac{1}{n^2}} = 1 < 1.1.$$

Thus there exists a constant A, s.t.

$$\frac{1}{n^2 - n} \leq \frac{1.1}{n^2} \tag{6}$$

if $n \geq A$.

The comparison test

In other words,

$$\frac{1}{n^2 - n} \leq \frac{1.1}{n^2} \tag{7}$$

if n is big enough.

Note $\displaystyle\sum_{n=1}^{\infty} \frac{1.1}{n^2}$ converges. Therefore by comparison test $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 - n}$ converges.

- We can also use the limit comparison test, which I will introduce after Example 5.