AS.110.109: Calculus II (Eng)

Review Session Midterm 2

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Polar Coordinates

Polar Coordinates:

Given any point $P = (x, y)$ on the plane

- r stands for the distance from the origin $(0, 0)$.
- θ stands for the angle from positive x-axis to OP.

Polar coordinate: (r, θ)

Polar Coordinates

- One can easily convert Cartesian coordinate (x, y) to polar coordinates (r, θ), and the other way around.

  r is a function of x, y;

  θ is also a function of x, y.

$$r = \sqrt{x^2 + y^2};$$

$$\theta = \tan^{-1} \frac{y}{x}.$$

Polar Coordinates

x is a function of r and θ;

y is a function of r and θ.

$$x = r\cos\theta;$$

$$y = r\sin\theta.$$

Areas and lengths in polar coordinates

Let $r = f(\theta)$ for $\theta \in [a, b]$ be a curve in the plane. The polar region is the region enclosed by the ray $\theta = a$, $\theta = b$ and the curve $r = f(\theta)$ for $\theta \in [a, b]$.

How to compute the area?

$$A = \int_a^b \frac{1}{2} r^2 d\theta = \int_a^b \frac{1}{2} f^2(\theta) d\theta.$$

Areas and lengths in polar coordinates

Arc length in polar curve $r = f(\theta)$ for $\theta \in [a, b]$

$$L = \int_a^b \sqrt{(f'(\theta))^2 + (f(\theta))^2} d\theta.$$

Improper Integrals: Part 1

The first type of improper integral:

$$\int_a^\infty f(x)dx := \lim_{b\to\infty} \int_a^b f(x)dx$$

if the limit exists ("exists" means "limit exists as a finite number").

- The improper integral $\int_a^\infty f(x)dx$ is convergent if the limit exists.

  It is divergent if the limit does not exist.

Improper Integrals: Part 1

- Geometric meaning:
  $\int_a^b f(x)dx$ is the area under the graph of $f(x)$ from a to b.
  $\int_a^\infty f(x)dx$ is the area under the graph of $f(x)$ from a to ∞.

Improper Integrals: Part 1

- Conclusion: $\int_1^\infty \frac{1}{x^p} dx$ converges if $p > 1$;
  $\int_1^\infty \frac{1}{x^p} dx$ diverges if $p \leq 1$.

Improper Integrals: Part 2

The second type of improper integral: the interval is finite, but the integrand is discontinuous at some points.

- If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t \to b-} \int_a^t f(x)dx$$

if the limit exists as a finite number.

Improper Integrals: Part 2

- If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t \to a+} \int_t^b f(x)dx$$

if the limit exists as a finite number.

Improper Integrals: Part 2

Therefore $\int_0^1 \frac{1}{x^p} dx$ converges when $p < 1$, $\int_0^1 \frac{1}{x^p} dx$ diverges when $p \geq 1$.

Improper Integrals: Part 2

What if there are both vertical asymptotes and ∞?

- Find all vertical asymptotes, as well as ∞ to write the integral as the sum of several improper integrals. Then compute each improper integral.

Improper Integrals: Part 2

- Comparison test:

  Suppose f and g are continuous with $f(x) \geq g(x) \geq 0$, for $x \geq a$.

  - (1). If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.
  - (2). If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Improper Integrals: Part 2

- Comparison test:

  (3). If $\int_a^\infty |f(x)|dx$ is convergent, then $\int_a^\infty f(x)dx$ is convergent.

Sequences

- Definition: A sequence (of numbers) is a list of $\{a_1, a_2, a_3, \cdots\}$ linearly ordered by an index set I. I is just the set of positive integers.

  Other ways to write a sequence:

  $$\{a_n\}_{n=1}^{\infty},$$

  or

  $$\{a_n\}_{n\in\mathbb{Z}^+}.$$

Sequences

- We will put a lot of effort to understand asymptotic behavior of a sequence, namely as $n \to \infty$.

Definition

If $\lim_{n\to\infty} a_n$ exists, we say the sequence is convergent. Otherwise, we say the sequence is divergent.

Sequences

- Fact 1: Suppose $f(x)$ is a function so that $a_n = f(n)$ for all $n \geq 1$.
  If $\lim_{x\to\infty} f(x) = A$, then $\lim_{n\to\infty} a_n = A$.

Sequences

- Fact 2: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.
  - Example 5.
$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0.$$

Sequences

- Fact 3' (Squeezing Theorem): If $c_n \leq a_n \leq b_n$, $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$, for some real number L, then $\lim_{n\to\infty} a_n = L$.

Sequences

- Geometric sequence

  $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$.

  $\lim_{n\to\infty} r^n = \infty$ if $|r| > 1$.

  $\lim_{n\to\infty}(-1)^n$ does not exist.

  $\lim_{n\to\infty} 1^n = 1$.

  Conclusion: The sequence $\{r^n\}$ converges if $-1 < r \leq 1$, and it diverges for all other values of r.

Sequences

- Fact 4: Every bounded, monotonic sequence is convergent.

Series

Definition

$a_1 + a_2 + a_3 + \cdots$ is called an infinite series or just series.

Denoted by

$$\sum_{n=1}^{\infty} a_n, or \sum a_n.$$

Series

Given a series $\sum_{n=1}^{\infty} a_n$, let s_n denote its partial sum

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

If $\lim_{n\to\infty} s_n$ exists as a finite number, then the series

$$\sum_{n=1}^{\infty} a_n := \lim_{n\to\infty} s_n,$$

and we say it is convergent.

If $\lim_{n\to\infty} s_n$ does not exist, we say $\sum_{n=1}^{\infty} a_n$ is divergent.

Series

- Example 1. Suppose $s_n = \frac{3n}{2n+3}$. Then $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{3n}{2n+3} = \frac{3}{2}$.

Series

- Example 2. Geometric series $a_n = a \cdot r^n$. Compute $\sum_{n=0}^{\infty} a_n$ for $-1 < r < 1$.
  - Remark: in other words, $\{a_n\}$ is a geometric series if $\frac{a_{n+1}}{a_n}$ equals r for all n.

  Solution:

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{1-r^n}{1-r}, \qquad (1)$$

for $r \neq 1$.

Series

If $r = -1$, then limit of $s_n = a\frac{1-r^n}{1-r}$ does not exist.
If $r = 1$, then the partial sum s_n is not equal to $a\frac{1}{1-r}$. It should be $s_n = na$ whose limit is infinity.
If $r \leq -1$ or $r > 1$, then limit of r^n does not exist. Hence limit of s_n does not exist.

Series

Conclusion: If $-1 < r < 1$, then

$$a + ar + ar^2 + \cdots = \sum_{n=0}^{\infty} ar^n = a\frac{1}{1-r}.$$

The series converges.

If $r \leq -1$ or $r \geq 1$, then $\sum_{n=0}^{\infty} ar^n$ diverges.

Series

- Work on some examples!

Series

Theorem

If $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

- Remark: The converse is not true in general.

Series

Contrapositive Statement of the Theorem: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

- Example. Determine if the series $\displaystyle\sum_{n=1}^{\infty} arctan\sqrt{n}$ is convergent or divergent.

Solution: Note $\displaystyle\lim_{n\to\infty} arctan\sqrt{n} = \frac{\pi}{2} \neq 0$. Thus by the Test for Divergence $\displaystyle\sum_{n=1}^{\infty} arctan\sqrt{n}$ diverges.