Representation of functions as power series

Function $\frac{1}{1-x}$ can be written as a power series (geometric series):

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$$

for $|x| < 1$.

Why $|x| < 1$?

Because $\frac{1}{1-x}$ can be written as a power series only when the series is convergent. The geometric series $\sum_{n=0}^{\infty} x^n$ is convergent on $|x| < 1$.

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- Example 1. Write $\frac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence.

  Solution: Replacing x by $-x^2$, we have

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

  It converges when $|-x^2| < 1$ that is $|x| < 1$. Of course, we could have determined the interval of convergence by ratio test, but it is unnecessary here.

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- Example 2. Write $\frac{1}{x+2}$ as the sum of a power series and find the interval of convergence.

  Solution:

$$\frac{1}{2+x} = \frac{1}{2(1-(-\frac{x}{2}))} = \frac{1}{2}\sum_{n=0}^{\infty}(-\frac{x}{2})^n$$

  It converges when $|-\frac{x}{2}| < 1$ that is $|x| < 2$. (Of course, we could have determined the interval of convergence by ratio test, but it is unnecessary here.)

Representation of functions as power series

How to take differentiation/integration of a function represented by a power series on the interval of convergence?

Take differentiation/integration term by term in the power series.

- Theorem If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

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(i)

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

(ii)

$$\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$$

The radius of convergence of the power series are both R.

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- Example 3. Express $\frac{1}{(1-x)^2}$ as a power series.

  Solution: $\frac{1}{(1-x)^2} = (\frac{1}{1-x})'$. Thus we can take the derivative term by term in the following identity

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n.$$

  and get

$$\frac{1}{(1-x)^2} = (\frac{1}{1-x})' = \sum_{n=1}^{\infty} nx^{n-1}.$$

  The radius of convergence is the same as for the original series.

  Radius of convergence is $R = 1$.

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- Example 4. Express $\ln(1+x)$ as a power series.

  Solution: $\ln(1+x) = \int \frac{1}{1+x} dx$.

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n.$$

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Taking the integration term by term, we get

$$\ln(1+x) = \int \frac{1}{1+x}dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} + C \\ = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} + C. \tag{7}$$

The radius of convergence is the same as for the original series.

Radius of convergence is $R = 1$.

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■ Example 5. Find the power series for $f(x) = \tan^{-1} x$.

Solution:

$$\begin{aligned} \tan^{-1} x &= \int \frac{1}{1+x^2} dx \\ &= \int \sum_{n=0}^{\infty} (-x^2)^n dx \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C. \end{aligned} \tag{8}$$

To find C, we put $x = 0$ and obtain $C = \tan^{-1} 0 = 0$.

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Thus

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

The radius of convergence is the same as for the original series.

Radius of convergence is $R = 1$.