Example 11. Determine if the series \( \sum_{n=1}^{\infty} \frac{\ln n + a \cdot n^{3/2}}{n^2} \) is convergent or divergent.

Solution: Use the limit comparison test:

\[
\lim_{n \to \infty} \frac{\ln n + a \cdot n^{3/2}}{n^2} = a
\]

if \( a \neq 0 \).

Also the series \( \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \) diverges. Thus the series \( \sum_{n=1}^{\infty} \frac{\ln n + a \cdot n^{3/2}}{n^2} \) diverges when \( a \neq 0 \).
The comparison test

For $a = 0$, then there exists a constant $A > 0$, s.t.

$$\frac{\ln n}{n^2} \leq \frac{1}{n^{1.9}}$$

for $n \geq A$. The series $\sum_{n=1}^{\infty} \frac{1}{n^{1.9}}$ converges. Thus $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges.
The comparison test

Remark: Note

\[
\lim_{n \to \infty} \frac{\ln n + an^{3/2}}{n^2} = 0
\]

But the limit comparison test doesn’t work when the series to be compared is divergent and the limit is 0.
Limit comparison test revisited:

The limit comparison test: suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

Suppose

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c.$$  

a) If $c > 0$ (including $c = \infty$), and $\sum b_n$ diverges, then $\sum a_n$ diverges.

b) If $c < \infty$ (including $c = 0$), and $\sum b_n$ converges, then $\sum a_n$ converges.
The comparison test

Proof (of the convergence case) of this modified version of limit comparison test can be seen from the following example. Proof (of the divergence case) of this modified version of limit comparison test can be derived similarly.

Example 12. Determine if the series $\sum_{n=1}^{\infty} \frac{n^3}{e^n}$ is convergent or divergent.

Solution: Compare $\frac{n^3}{e^n}$ with $\frac{2^n}{e^n}$. Since

$$\lim_{n \to \infty} \frac{n^3}{2^n} = 0,$$

there exists a constant $A$, s.t. for $n \geq A$,

$$\frac{n^3}{2^n} < 1.$$
Hence for $n \geq A$,

$$\frac{n^3}{e^n} = \frac{n^3}{2^n} \cdot \frac{2^n}{e^n} < \frac{2^n}{e^n}.$$ 

Since the series $\sum_{n=1}^{\infty} \frac{2^n}{e^n}$ converges, by comparison test $\sum_{n=1}^{\infty} \frac{n^3}{e^n}$ also converges.
Example 13. Determine if the series \( \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}} \) is convergent or divergent.

Solution: Compare \( \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}} \) with \( \frac{1}{n} \).

Note

\[
\frac{1}{n^{1+\frac{1}{n}}} = \frac{1}{n} \cdot \frac{1}{\sqrt[n]{n}},
\]

and

\[
\lim_{n \to \infty} \frac{1}{\sqrt[n]{n}} = 1.
\]
The comparison test

We have

\[
\lim_{n \to \infty} \frac{1}{\frac{n^{1+\frac{1}{n}}}{\frac{1}{n}}} = 1
\]

It is known that \( \frac{1}{n} \) diverges. Thus \( \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}} \) also diverges.
Example 14. Determine if the series \( \sum_{n=1}^{\infty} \frac{n!}{n^n} \) is convergent or divergent.

Solution: Let’s say \( n \) is even for example. We can write

\[
\frac{n!}{n^n} = \frac{1 \cdot 2 \cdots \frac{n}{2}}{n \cdot \frac{n}{2} \cdots \frac{n}{2}} \cdot \frac{\left(\frac{n}{2} + 1\right)}{n} \cdots \frac{n}{n}
\]

Now each term \( \frac{i}{n} \leq \frac{1}{2} \) for \( i = 1, \cdots, \frac{n}{2} \). Also each term \( \frac{i}{n} \leq 1 \) for \( i = \frac{n}{2} + 1, \cdots, n \).

Thus

\[
\left[\frac{1}{n} \cdot \frac{2}{n} \cdots \frac{n}{n}\right] \cdot \left[\frac{\left(\frac{n}{2} + 1\right)}{n} \cdots \frac{n}{n}\right] \leq \left[\frac{1}{2} \cdots \frac{1}{2}\right] \cdot [1 \cdots 1] = \left(\frac{1}{2}\right)^{n/2}
\]
The comparison test

\[ \sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^{n/2} = \sum_{n=1}^{\infty} \left( \frac{1}{4} \right)^n \text{ converges. Thus } \sum_{n=1}^{\infty} \frac{n!}{n^n} \text{ converges.} \]