Root test

Root test

Consider the limit $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$, suppose it exists.

- $L < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ is absolutely convergent (thus convergent);
- $L > 1$ (including $L = \infty$) $\Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent
- $L = 1 \Rightarrow$ the test is inconclusive.

Root test

- Example 5. Determine whether

$$\sum_{n=1}^{\infty}(\frac{-2n}{n+1})^{5n}$$

is convergent or divergent.

Solution:

$$\sqrt[n]{|a_n|} = (\frac{2n}{n+1})^5 \to 2^5 > 1$$

Thus the series $\sum_{n=1}^{\infty}(\frac{-2n}{n+1})^{5n}$ diverges.

Root test

- Example 6. Determine whether

$$\sum_{n=1}^{\infty}(\frac{-2n}{n+1})^{5n}\cdot n$$

is convergent or divergent.

Solution:

$$\sqrt[n]{|a_n|}=\sqrt[n]{n}(\frac{2n}{n+1})^5.$$

- Fact:

$$\lim_{n\to\infty}\sqrt[n]{n}=1.$$

So

$$\lim_{n\to\infty}\sqrt[n]{n}(\frac{2n}{n+1})^5=2^5>1$$

Thus the series $\sum_{n=1}^{\infty}(\frac{-2n}{n+1})^{5n}$ diverges.

Power series

A power series is a series of the form

$$\sum_{n=1}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \cdots$$

For each value of x we can test for convergence or divergence. It may converge for some values of x and diverge for other values of x.

Power series

The sum of the series is a function

$$f(x) = \sum_{n=1}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \cdots$$

The domain of this function is the set of all x for which the series converges. f has infinitely many terms.

Power series

Geometric series is a special kind of power series

$$\sum_{n=1}^{\infty} x^n = 1 + x + x^2 + x^3 \cdots$$

It converges when $-1 < x < 1$ and diverges when $|x| \geq 1$. The function $f(x)$ (when the series converges) equals

$$f(x) = \frac{1}{1-x}$$

Power series

- Example 1. For what values of x does the series

$$\sum_{n=1}^{\infty} n!x^n$$

converge?

  - Solution: Ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty}(n+1)|x| = \infty$$

if $x \neq 0$.

By ratio test, the series diverges when $x \neq 0$.

Thus the series converges only when $x = 0$.

Power series

- Example 2. For what values of x does the series

$$\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^{1/2}}$$

converge?

  - Solution: Root test

$$\sqrt[n]{\left|\frac{(x-5)^n}{n^{1/2}}\right|} = \frac{|x-5|}{(\sqrt[n]{n})^{1/2}}$$

We know that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Power series

Thus

$$\lim_{n\to\infty} |\frac{x-5}{(\sqrt[n]{n})^{1/2}} = |x-5|.$$

By root test,

1) when $4 < x < 6$, the series converges

2) when $|x-5| > 1$, the series diverges

Power series

When $x = 4$, the root test is inconclusive. The series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}$. By the alternating series test, the series converges. When $x = 6$, the root test is inconclusive. The series becomes $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$. This is a divergent p-series (for $p = 1/2$).

Power series

Theorem

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$*, there are only three possibilities:*

(i) The series converges only when $x = a$*;*

(ii)The series converges for all x*;*

(iii)There is a positive number R *such that the series converges if* $|x-a| < R$ *and diverges if* $|x-a| > R$*.*

Power series

R in case (iii) is called the radius of convergence.

In case (i), radius of convergence equals 0.

In case (ii), radius of convergence equals ∞.

Power series

Example 3.

$$\sum_{n=0}^{\infty} c_n(x-5)^n$$

is convergent at $x = 7$. Can we say anything at $x = 4$?

Solution: Since the series is convergent at $x = 7$, the radius of convergence $R \geq 2$ (because $7 - 5 = 2$). Thus the series must be convergent in between $(5 - 2, 5 + 2) = (3, 7)$. $x = 4$ lies inside this interval. Thus the series converges at $x = 4$.

Power series

In the above theorem, it does not specify if the series converges or diverges at the end points of the interval $(a - R, a + R)$.
What happens at $a - R$ and $a + R$?
In fact, anything can happen at the endpoints. In other words, the series may diverge at one or both endpoints; and it may converge at one or both endpoints.

Power series

More precisely, the interval of convergence could be any one of the following four cases:

$(a-R, a+R), [a-R, a+R], (a-R, a+R]$, or $[a-R, a+R)$.

Power series

- Example 4. For what values of x does the series

$$\sum_{n=1}^{\infty} n!x^n$$

converge?

  - Solution: Ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty}(n+1)|x| = \infty$$

if $x \neq 0$.

By ratio test, the series diverges when $x \neq 0$.

Thus the series converges only when $x = 0$.

Radius of convergence $R = 0$. The interval of convergence is

$x = 0$.

Power series

- Example 5. For what values of x does the series

$$\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^{1/2}}$$

converge?

Solution:

We discussed in Example 2 we can use Root test

$$\sqrt[n]{\left|\frac{(x-5)^n}{n^{1/2}}\right|} = \frac{|x-5|}{(\sqrt[n]{n})^{1/2}}$$

We know that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Power series

Thus

$$\lim_{n\to\infty} |\frac{x-5}{(\sqrt[n]{n})^{1/2}} = |x-5|.$$

By root test,

1) when $4 < x < 6$, the series converges

2) when $|x-5| > 1$, the series diverges

Radius of convergence $R = 1$. When $x = 4$, the series converges by the alternating series test.

When $x = 6$, the series diverges. The interval of convergence is $[4, 6)$.