Ratio test

Ratio test

Consider the limit $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, suppose it exists.

- $L < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ is absolutely convergent (thus convergent);
- $L > 1$ (including $L = \infty$) $\Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent
- $L = 1 \Rightarrow$ the test is inconclusive.

Ratio test

- Example 4. Determine whether

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{2^n}$$

is convergent or divergent.

Solution: Compute the ratio

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}\frac{(n+1)^3}{2^{n+1}}}{(-1)^n\frac{n^3}{2^n}}\right| = \frac{1}{2}\frac{(n+1)^3}{n^3}$$

Ratio test

Let $n \to \infty$.

$$\lim_{n\to\infty} \frac{1}{2}\frac{(n+1)^3}{n^3} = \frac{1}{2} < 1$$

Thus by the ratio test, the series is absolutely convergent and thus is convergent.

Ratio test

- Example 5. Determine whether

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

  is convergent or divergent.

  We have discussed a similar example when learning Comparison Test.

- Solution by Comparison Test: We can write

$$\frac{n^n}{n!} = \frac{n}{1} \cdot \frac{n}{2} \cdots \frac{n}{\frac{n}{2}} \cdot \frac{n}{(\frac{n}{2}+1)} \cdots \frac{n}{n}$$

  Now each term $\frac{n}{i} \geq 2$ for $i = 1, \cdots \frac{n}{2}$. Also each term $\frac{n}{i} \geq 1$ for $i = \frac{n}{2} + 1, \cdots n$.

Ratio test

Thus we break the products into two parts and use these estimates

$$[\frac{n}{1}\cdot\frac{n}{2}\cdots\frac{n}{\frac{n}{2}}]\cdot[\frac{n}{(\frac{n}{2}+1)}\cdots\frac{n}{n}]\geq[2\cdots 2]\cdot[1\cdots 1]=2^{n/2}$$

Since $\sum_{n=1}^{\infty}(2)^{n/2}=\sum_{n=1}^{\infty}(\sqrt{2})^{n}$ diverges, by comparison test $\sum_{n=1}^{\infty}\frac{n^n}{n!}$ diverges.

Ratio test

- Solution using the Ratio Test: Compute the ratio

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n!}{n^n}}\right| = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$

Ratio test

Let $n \to \infty$.

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = e > 1$$

Thus by the ratio test, the series is divergent.