The integral test and estimates of sums

- Example 1. Determine if $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent or divergent.

  Solution:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

Since $\frac{1}{x^2}$ is decreasing,

$$\frac{1}{2^2} \leq \int_1^2 \frac{1}{x^2} dx,$$

$$\frac{1}{3^2} \leq \int_2^3 \frac{1}{x^2} dx,$$

$$\frac{1}{4^2} \leq \int_3^4 \frac{1}{x^2} dx, \cdots$$

The integral test and estimates of sums

$$
\begin{aligned}
&1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots \\
\le &1+\int_1^2 \frac{1}{x^2}\,dx+\int_2^3 \frac{1}{x^2}\,dx+\int_3^4 \frac{1}{x^2}\,dx+\cdots \qquad (3)\\
= &1+\int_1^\infty \frac{1}{x^2}\,dx.
\end{aligned}
$$

The integral test and estimates of sums

Since $\int_1^\infty \frac{1}{x^2} dx$ is convergent, the partial sum s_n is bounded. Also, $a_n > 0$ implies s_n is increasing.

- Recall Fact 5: Every bounded monotonic sequence is convergent.

Thus s_n's limit exists, i.e. $\sum_{n=1}^\infty \frac{1}{n^2}$ is convergent.

The integral test and estimates of sums

This example shows us that the convergence of $\int_1^\infty \frac{1}{x^2}dx$ implies convergence of the series $\sum_{n=1}^\infty \frac{1}{n^2}$.

This can be generalized to a large class of series:

The integral test and estimates of sums

The integral test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent.

The integral test and estimates of sums

- Example 2. For what values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?
  Solution: Use the integral test. We know from Chapter 7.8 Indefinite Integral that $\int_1^{\infty} \frac{1}{x^p} dx$ is convergent if $p > 1$, and divergent if $p \leq 1$.
  Conclusion: The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$, and divergent if $p \leq 1$.

The integral test and estimates of sums

- Example 3. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^3+4}$ converges or diverges.
  - Solution: Use the integral test.

Consider $f(x) = \frac{1}{x^3+4}$. It is positive, continuous and decreasing. Why is it decreasing? Because $x^3 + 4$ is increasing.

The integral test and estimates of sums

Now

$$\int_1^\infty \frac{1}{x^3+4}dx \leq \int_1^\infty \frac{1}{x^3}dx.$$

By Chapter 7.8, the indefinite integral on the right hand side is convergent. Thus $\int_1^\infty \frac{1}{x^3+4}dx$ is also convergent. So the series is convergent.