Alternating series

- Example 6.

$$\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{2n+3}$$

  converges.

  - Solution: It is obvious

$$\lim_{n\to\infty}\frac{\sqrt{n}}{2n+3}=0.$$

  Thus all we need (in order to apply the alternating series test) is $\frac{\sqrt{n}}{2n+3}$ is decreasing.

Alternating series

Consider $f(x) = \frac{\sqrt{x}}{2x+3}$. By computation, $f'(x) = \frac{\frac{3}{2}-x}{\sqrt{x}(2x+3)^2}$. Thus $f'(x) < 0$ is $x > \frac{3}{2}$. In particular, $b_2 > b_3 > b_4 \cdots$. Thus the series is convergent.

- Remark: It is enough to assume b_n is decreasing for n big enough (i.e. there exists an integer $A > 0$, such that b_n is decreasing for $n \geq A$).

Alternating series

- Example 7. Let f and g are two polynomials

$$\sum_{n=1}^{\infty}(-1)^n \frac{f(n)}{g(n)}$$

converges if and only if $deg(f) < deg(g)$.

  - Solution: All we need is $\frac{f(n)}{g(n)}$ is decreasing for big enough n, and

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

When $deg(f) < deg(g)$ holds, the above two statements are true.

Thus by the alternating series test, it is convergent.

Alternating series

If $deg(f) \geq deg(g)$, then $\lim_{n\to\infty} \frac{f(n)}{g(n)} \neq 0$. Thus by the divergence test, the series diverges.

Absolute convergence series

Definition

A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Absolute convergence series

If all a_n are positive ($a_n \geq 0$), then absolute convergence is the same as convergence. But in general,

$$-\sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} |a_n|$$

Absolute convergence series

Theorem

If a series $\sum a_n$ *is absolutely convergent, then it is convergent.*

Absolute convergence series

- Example 1.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2+2n}$$

▶ Note $\sum_{n=1}^{\infty}\frac{1}{n^2+2n}$ converges, by limit comparison test with series $\sum_{n=1}^{\infty}\frac{1}{n^2}$.
Thus using Theorem, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2+2n}$ converges.

Absolute convergence series

Definition

A series is called conditionally convergent if it is convergent but not absolutely convergent.

- Example 2.

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^p}$$

for $0 < p \leq 1$ are conditionally convergent.

Absolute convergence series

- Solution: We note by using the alternating series test, $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^p}$ converges. But the p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges for these values of p.

  Thus $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^p}$ is conditionally convergent.

Absolute convergence series

- Example 3. Determine whether

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$$

is convergent or divergent.

Solution:

$$\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right| \leq \sum_{n=1}^{\infty} \left|\frac{1}{n^2}\right|$$

Absolute convergence series

and we know

$$\sum_{n=1}^{\infty} \left|\frac{1}{n^2}\right|$$

converges. Thus

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$$

is absolutely convergent and therefore is convergent.