The comparison test

Limit comparison test revisted:

The limit comparison test: suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

Suppose

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c.$$

a) If $c > 0$ (including $c = \infty$), and $\sum b_n$ diverges, then $\sum a_n$ diverges.

b) If $c < \infty$ (including $c = 0$), and $\sum b_n$ converges, then $\sum a_n$ converges.

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- Proof (of the convergence case) of this modified version of limit comparison test can be seen from the following example. Proof (of the divergence case) of this modified version of limit comparison test can be derived similarly.
  - Example 12. Determine if the series $\sum_{n=1}^{\infty} \frac{n^3}{e^n}$ is convergent or divergent.

Solution: Compare $\frac{n^3}{e^n}$ with $\frac{2^n}{e^n}$. Since

$$\lim_{n\to\infty} \frac{n^3}{2^n} = 0,$$

there exists a constant A, s.t. for $n \geq A$,

$$\frac{n^3}{2^n} < 1.$$

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Hence for $n \geq A$,

$$\frac{n^3}{e^n} = \frac{n^3}{2^n} \cdot \frac{2^n}{e^n} < \frac{2^n}{e^n}.$$

Since the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{e^n}$ converges, by comparison test $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{e^n}$ also converges.

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- Example 13. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ is convergent or divergent.

Solution: Compare $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ with $\frac{1}{n}$.

Note

$$\frac{1}{n^{1+\frac{1}{n}}} = \frac{1}{n}\frac{1}{\sqrt[n]{n}},$$

and

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1.$$

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We have

$$\lim_{n\to\infty} \frac{\frac{1}{n^{1+\frac{1}{n}}}}{\frac{1}{n}} = 1$$

It is known that $\frac{1}{n}$ diverges. Thus $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ also diverges.

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▶ Example 14. Determine if the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is convergent or divergent.

Solution: Let's say n is even for example. We can write

$$\frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdots \frac{\frac{n}{2}}{n} \cdot \frac{(\frac{n}{2}+1)}{n} \cdots \frac{n}{n}$$

Now each term $\frac{i}{n} \leq \frac{1}{2}$ for $i = 1, \cdots \frac{n}{2}$. Also each term $\frac{i}{n} \leq 1$ for $i = \frac{n}{2} + 1, \cdots n$.

Thus

$$[\frac{1}{n} \cdot \frac{2}{n} \cdots \frac{\frac{n}{2}}{n}] \cdot [\frac{(\frac{n}{2}+1)}{n} \cdots \frac{n}{n}] \leq [\frac{1}{2} \cdots \frac{1}{2}] \cdot [1 \cdots 1] = (\frac{1}{2})^{n/2}$$

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$$\sum_{n=1}^{\infty}(\frac{1}{2})^{n/2} = \sum_{n=1}^{\infty}(\frac{1}{\sqrt{2}})^{n} \text{ converges. Thus } \sum_{n=1}^{\infty}\frac{n!}{n^n} \text{ converges.}$$