# The comparison test

- Example 5. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{3^n - 2}$ is convergent or divergent.

  Remark: We cannot use the same method as Example 1.

$$\frac{1}{3^n - 2} > \frac{1}{3^n}$$

  is useless as far as the Comparison test because $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is convergent.

# The comparison test

The limit comparison test Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where $c$ is a finite number, $c > 0$ then either both series converge or both diverge.

Remark: The limit $c \neq 0$.

# The comparison test

- Example 7. Determine if the series $\sum_{n=1}^{\infty} \frac{n+3}{\sqrt{n+5}}$ is convergent or divergent.

  Solution: Diverges, since

$$\lim_{n\to\infty} \frac{\frac{n+3}{\sqrt{n+5}}}{\frac{n}{\sqrt{n}}} = 1,$$

  and the series $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n}}$ diverges.

# The comparison test

- Example 8. Determine if the series $\sum_{n=1}^{\infty} \frac{e^n}{n^8}$ is convergent or divergent.

  Solution: Note $\lim_{n\to\infty} \frac{e^n}{n^8} = \infty \neq 0$, thus by the divergence test $\sum_{n=1}^{\infty} \frac{e^n}{n^8}$ diverges.

# The comparison test

- Example 9. Determine if the series $\sum_{n=1}^{\infty} arctan\sqrt{n}$ is convergent or divergent.

  Solution: Note $\lim_{n\to\infty} arctan\sqrt{n} = \frac{\pi}{2} \neq 0$, thus by the divergence test $\sum_{n=1}^{\infty} arctan\sqrt{n}$ diverges.

# The comparison test

■ Example 10. Determine if the series $\sum_{n=1}^{\infty} \frac{2n^3+3}{n^4}$ is convergent or divergent.

Solution: Use the limit comparison test:

$$\lim_{n\to\infty} \frac{\frac{2n^3+3}{n^4}}{\frac{2n^3}{n^4}} = 1$$

Also the series $\sum_{n=1}^{\infty} \frac{2}{n}$ diverges. Thus the series $\sum_{n=1}^{\infty} \frac{2n^3+3}{n^4}$ diverges.

# The comparison test

- Example 11. Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n + a \cdot n^{3/2}}{n^2}$ is convergent or divergent.

  Solution: Use the limit comparison test:

  $$\lim_{n\to\infty} \frac{\frac{\ln n + a n^{3/2}}{n^2}}{\frac{1}{n^{1/2}}} = a$$

  if $a \neq 0$.

  Also the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges. Thus the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n + a \cdot n^{3/2}}{n^2}$ diverges when $a \neq 0$.

# The comparison test

For $a = 0$, then there exists a constant $A > 0$, s.t.

$$\frac{\ln n}{n^2} \leq \frac{1}{n^{1.9}}$$

for $n \geq A$. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.9}}$ converges. Thus $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges.

# The comparison test

- Remark: Note
$$\lim_{n\to\infty} \frac{\frac{\ln n + a n^{3/2}}{n^2}}{\frac{1}{n^{1/2}}} = 0$$
But the limit comparison test doesn't work when the series to be compared is divergent and the limit is 0.