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Alternating series

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■ Example 1.

\[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots\]
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An alternating series is a series whose terms are alternately positive and negative.

Example 1.

\[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots \]

Example 2.

\[-1 + 1 - 1 + 1 - 1 \ldots \]
More precisely, alternating series takes the form that

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n \quad \text{(e.g. Example 1)}$$

or

$$\sum_{n=1}^{\infty} (-1)^n b_n \quad \text{(e.g. Example 2)}$$

where $b_n > 0$. 
Alternating series

**Alternating series test:** If an alternating series satisfies

1) $b_{n+1} \leq b_n$ for all $n$,

2) $\lim_{n \to \infty} b_n = 0$,

then the series converges.
Alternating series

The series of Example 1 converges.
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Solution:

\[
\frac{1}{n} > 0; \\
\frac{1}{n + 1} < \frac{1}{n};
\]

and

\[
\lim_{n \to \infty} \frac{1}{n} = 0.
\]

It satisfies the conditions in alternating series test. Thus

\[
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots
\]

converges.
Alternating series

Remark: This is different from harmonic series

\[
\sum_{n=1}^{\infty} \frac{1}{n},
\]

which is divergent.
Example 3.

\[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2 + 2n} \]

Solution: This is also an alternating series in which \( b_n \) is decreasing and tends to 0. By the alternating series test, it is convergent.
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Solution: This is also an alternating series in which \( b_n \) is decreasing and tends to 0. By the alternating series test, it is convergent.
Note $\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$ converges, by limit comparison test with series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Fact: Suppose $b_n > 0$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} (-1)^n b_n$ also converges. But it is not the other way around.
Example 4.

\[
\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}
\]

Solution: This is an alternating series in which \(b_n\) is decreasing and tends to 0. By the alternating series test, it is convergent.
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Example 5. For all $p > 0$,

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Solution: This is an alternating series in which $b_n$ is decreasing and tends to 0. By the alternating series test, it is convergent.
Example 6.

\[ \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{2n + 3} \]

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Solution: It is obvious

\[ \lim_{n \to \infty} \frac{\sqrt{n}}{2n + 3} = 0. \]

Thus all we need (in order to apply the alternating series test) is \( \frac{\sqrt{n}}{2n+3} \) is decreasing.
Consider $f(x) = \frac{\sqrt{x}}{2x+3}$. By computation, $f'(x) = \frac{3\sqrt{x} - x}{\sqrt{x}(2x+3)^2}$. Thus $f'(x) < 0$ if $x > \frac{3}{2}$. In particular, $b_2 > b_3 > b_4 \cdots$. Thus the series is convergent.

Remark: It is enough to assume $b_n$ is decreasing for $n$ big enough (i.e. there exists an integer $A > 0$, such that $b_n$ is decreasing for $n \geq A$).
Example 7. Let $f$ and $g$ are two polynomials

$$\sum_{n=1}^{\infty} (-1)^n \frac{f(n)}{g(n)}$$

converges if and only if $\text{deg}(f) < \text{deg}(g)$.

Solution: All we need is $\frac{f(n)}{g(n)}$ is decreasing for big enough $n$, and

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0.$$

When $\text{deg}(f) < \text{deg}(g)$ holds, the above two statements are true. Thus by the alternating series test, it is convergent.
Alternating series

If \( \text{deg}(f) \geq \text{deg}(g) \), then \( \lim_{n \to \infty} \frac{f(n)}{g(n)} \neq 0 \). Thus by the divergence test, the series diverges.