Math 109, Fall 2018
Midterm 2

Solution

Name:
Section:

Requirements:

■This exam should be completed in **45 minutes**.

■Books, notes, calculators, computers, discussion and collaboration are not allowed.

■Do all of your work in this exam booklet.

■Simplify all answers as far as possible.

■Solutions without proper justification will receive no credit.

Problem	**Points**	**Score**
1	15	
2	15	
3	20	
4	20	
5	10	
6	20	
Total	100	

Problem 1. (15') Find the area of the region that is bounded by the polar curve $r = \sin\theta$ and lies in the sector $\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$.

$$\text{Area} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2} \sin^2\theta \; d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{4} (1 - \cos 2\theta)\, d\theta$$

$$= \frac{1}{4}\theta - \frac{1}{8} \sin 2\theta \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{4} \frac{\pi}{6} - \frac{1}{8} \left(\sin \frac{2\pi}{3} - \sin \frac{\pi}{3} \right)$$

$$= \frac{1}{4} \frac{\pi}{6} - \frac{1}{8} \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \right)$$

$$= \frac{\pi}{24}.$$

Problem 2. (15') Determine whether the following improper integral converges or diverges. If it converges, compute it.

$$\int_0^1 \frac{1}{x^2-4x+3}dx.$$

$$\frac{1}{x^2-4x+3} = \frac{1}{(x-3)(x-1)} = \frac{A}{x-3} + \frac{B}{x-1}$$

$$A(x-1) + B(x-3) = 1$$

$$\Rightarrow \begin{cases} A+B = 0 \\ -A-3B = 1 \end{cases}$$

$$\Rightarrow A = 1/2 , \quad B = -1/2$$

$$\Rightarrow \quad \int_0^1 \frac{1\,dx}{x^2-4x+3} = \frac{1}{2}\int_0^1 \frac{1}{x-3}dx - \frac{1}{2}\int_0^1 \frac{1}{x-1}dx$$

$$= \frac{1}{2} \ln|x-3|\Big]_0^1 - \frac{1}{2}\lim_{b\to 1^-} \ln|x-1|\Big]_0^b$$

$$= \frac{1}{2}\Big(\ln|-2| - \ln|-3| - \underbrace{\lim_{b\to 1^-} \ln|b-1|}_{D.N.E.} + \ln|-1|\Big)$$

$$\Rightarrow \quad \int_0^1 \frac{dx}{x^2-4x+3} \quad \text{diverges}$$

Problem 3. (20') Determine whether the following improper integral converges or diverges. Explain it.

a) $\displaystyle\int_1^{10} \frac{1}{\sqrt[5]{x-3}}dx.$

b) $\displaystyle\int_{10}^{\infty} \frac{1}{x^2-4}dx.$

a) discontinuous at $x=3$

$$= \lim_{b\to 3^-}\int_1^b \frac{1}{\sqrt[5]{x-3}}\,dx + \lim_{b\to 3^+}\int_b^{10} \frac{1}{\sqrt[5]{x-3}} \qquad \text{Let } u=x-3$$

$$= \lim_{b\to 0^-}\int_{-2}^{0} \frac{1}{u^{\frac{1}{5}}}\,du + \lim_{b\to 0^+}\int_0^{10} \frac{1}{u^{\frac{1}{5}}}\,du$$

$\therefore$ convergent by p-test.

b) $\displaystyle \frac{1}{x^2-4} \le \frac{1}{0.9x^2}$

$\displaystyle\int_{10}^{\infty} \frac{1}{0.9x^2}\,dx$ is convergent by p-test,

so $\displaystyle\int_{10}^{\infty} \frac{1}{x^2-4}\,dx$ is convergent by comparison test.

Problem 4. (20') Determine whether the following sequence $\{a_n\}_{n=1}^{\infty}$ converges or diverges. If it converges, compute the limit.

a) $a_n = \dfrac{\ln(n^2+2)}{\ln(3n)}$.

Q4:

a. We have $\lim_{n\to\infty} = \dfrac{\ln(n^2+2)}{\ln(3n)} = \dfrac{\infty}{\infty}$, so using [redacted]

L-Hopital we obtain

$$\frac{\lim_{n\to\infty} \ln(n^2+2)}{\lim_{n\to\infty} \ln(3n)} = \frac{\frac{2n}{n^2+2}}{\frac{3}{3n}} = \frac{2n}{n^2+2}\,\frac{3n}{3} = \frac{6n^2}{3n^2+6}$$

[redacted]

$$\text{so } \lim_{n\to\infty} \frac{\ln(n^2+2)}{\ln(3n)} = \lim_{n\to\infty} \frac{6n^2}{3n^2+6} = 2$$

[redacted]

common mistakes

- computing $d/dn \ln(n^2+2)$ and $d/dn \ln(3n)$ wrong

[redacted]

- saying that $\dfrac{\ln(n^2+2)}{\ln(3n)} = \ln\left(\dfrac{n^2+2}{3n}\right)$ [redacted]
- saying that $\lim_{n\to\infty} \dfrac{\ln(n^2+2)}{\ln(3n)} = \lim_{n\to\infty} \dfrac{\ln(d/dn\,(n^2+2))}{\ln(d/dn\,(3n))}$

[redacted]

b) $a_n = \dfrac{(-1)^n}{2\sqrt{n}}$.

b. Since

$$\frac{-1}{2\sqrt{n}} \le \frac{(-1)^n}{2\sqrt{n}} \le \frac{1}{2\sqrt{n}}$$ [redacted]

and

$$\lim_{n\to\infty} \frac{-1}{2\sqrt{n}} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{2\sqrt{n}} = 0$$ [redacted]

we have that

$$\lim_{n\to\infty} \frac{(-1)^n}{2\sqrt{n}} = 0 \quad \text{by squeeze theorem.}$$

[redacted]

Common mistakes

- thinking that the sequence is geometric [redacted]
- computing $\lim_{n\to\infty} \frac{1}{2\sqrt{n}}$ and $\lim_{n\to\infty} \frac{-1}{2\sqrt{n}}$ wrong

[redacted]

- confusing the sequence with a series [redacted]

Problem 5. (10') Determine whether the series $\sum_{n=1}^{\infty} \cos \frac{1}{n^3}$ converges or diverges. Explain it.

Sol

$\cos \frac{1}{n^3} \longrightarrow \cos 0 = 1$ as $n \to \infty$

$1 \neq 0$ $\therefore \sum_{n=1}^{\infty} \cos \frac{1}{n^3}$ must diverge.

$\square$

$\boxed{\text{Recall:}}$ $\sum x_n$ converges $\Rightarrow \lim_{n\to\infty} x_n = 0$

i.e. If $\lim_{n\to\infty} x_n \neq 0$, then $\sum x_n$ must diverge.

Problem 6. (20') Determine whether the series converges or diverges. If it converges, compute it.

a) $\sum_{n=0}^{\infty} \frac{2}{5^{n+1}}$

Notice that the series is geometric.

a = first term = $\frac{2}{5^{0+1}} = \frac{2}{5}$

$$r = \frac{\frac{2}{5^{n+2}}}{\frac{2}{5^{n+1}}} = \frac{1}{5}$$

Since $|r| < 1$, series converges to

$$\frac{a}{1-r} = \frac{\frac{2}{5}}{1-\frac{1}{5}} = \frac{1}{2}$$

b) $\sum_{n=0}^{\infty}(\frac{3}{2})^n$

This is a geometric series w/ $r = \frac{3}{2}$.

Since $|r| \geq 1$, the series diverges.

OR

Since $\lim_{n \to \infty} (\frac{3}{2})^n = \infty$

By divergence test, series diverges.