Prob 1 a) Compute $\int \tan^5 x \sec^3 x \, dx$ observe: $d(\sec x) = \tan x \sec x \, dx$

sol:

$$= \int \tan^4 x \sec^2 x \, d(\sec x)$$

$$= \int (\sec^2 x - 1)^2 \sec^2 x \, d(\sec x)$$ let $u = \sec x$

$$= \int (u^2-1)^2 u^2 \, du$$

$$= \int (u^4 - 2u^2 + 1) u^2 \, du$$

$$= \int u^6 - 2u^4 + u^2 \, du$$

$$= \frac{1}{7} u^7 - \frac{2}{5} u^5 + \frac{1}{3} u^3 + C$$

$$= \frac{1}{7} \sec^7 x - \frac{2}{5} \sec^5 x + \frac{1}{3} \sec^3 x + C .$$

(b). $\int \sin^2 x \cos^2 x dx$.

$$= \int \frac{(1-\cos 2x)}{2} \frac{(1+\cos 2x)}{2} \, dx$$

$$= \frac{1}{4}\int (1-\cos^2 2x)\,dx = \frac{1}{4}\int \sin^2 2x \, dx$$

$$= \frac{1}{8}\int (1-\cos 4x)\, dx$$

$$= \frac{1}{8}\left(x - \frac{\sin 4x}{4}\right) + C$$

2. Solve the differential equation

$$y'(x) = (y-1)\cos x$$

$$\frac{dy}{dx} = (y-1)\cos x$$

Use separation of variables to obtain

$$\frac{dy}{y-1} = \cos x \, dx \qquad \text{if } y \neq 1$$

Then

$$\int \frac{dy}{y-1} = \int \cos x \, dx$$

$$\ln|y-1| = \sin x + C$$

$$y-1 = \pm e^{\sin x + C}$$

$$y = Ce^{\sin x} + 1$$

If $y=1$, we have that $(y-1)\cos x = 0$ and $dy/dx = 0$, so $y=1$ is another solution

Thus the solutions are

$$y = ce^{\sin x} + 1$$

and

$$y = 1$$

Problem 3. (15') Solve the differential equation

$$y'(x) + y = \sin x, y(0) = 1. \tag{2}$$

3. $y' + P(x)\, y = Q(x).$

Integrating factor $e^{\int P(x)\, dx} = e^{\int 1\, dx} = e^x.$

$e^x y'(x) + e^x \cdot y = e^x \sin x.$

$\int (e^x y)'\, dx = \int e^x \sin x\, dx$

$e^x y(x) = \int e^x \sin x\, dx.$

$$\begin{aligned}\int e^x \sin x\, dx &= e^x \sin x - \int e^x \cos x\, dx \\ &= e^x \sin x - \left(e^x \cos x - \int e^x (-\sin x)\, dx\right) \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x\, dx\end{aligned}$$

Thus $\int e^x \sin x\, dx = \dfrac{e^x \sin x - e^x \cos x}{2} + C$

$e^x y(x) = \dfrac{e^x \sin x - e^x \cos x}{2} + C$

$y(x) = \dfrac{\sin x - \cos x}{2} + \dfrac{C}{e^x}.$

$y(0) = 1 \Rightarrow 1 = \dfrac{\sin 0 - \cos 0}{2} + \dfrac{C}{e^0} = -\dfrac{1}{2} + C$

Thus $C = \dfrac{3}{2}.$ $y(x) = \dfrac{\sin x - \cos x}{2} + \dfrac{\frac{3}{2}}{e^x}.$

Problem 4. (20') Compute the following integral

$$\int \frac{e^{3x}}{e^{2x} - 2e^{x} + 1} dx. \tag{3}$$

Solution. Do substitution $t = e^x$

then $dt = e^x dt$

Then the integral $= \int \frac{e^{2x} \cdot e^x dx}{e^{2x} - 2e^x + 1} = \int \frac{t^2}{t^2 - 2t + 1} dt$.

$= \int \frac{t^2}{(t-1)^2} dt$.

$= \int 1 + \frac{2t-1}{(t-1)^2} dt$ $\qquad 2t-1 = (2t-2) + 1$.

$= \int 1 + \frac{2(t-1)+1}{(t-1)^2} dt = \int 1 \, dt + \int \frac{2}{t-1} + \int \frac{1}{(t-1)^2} dt$.

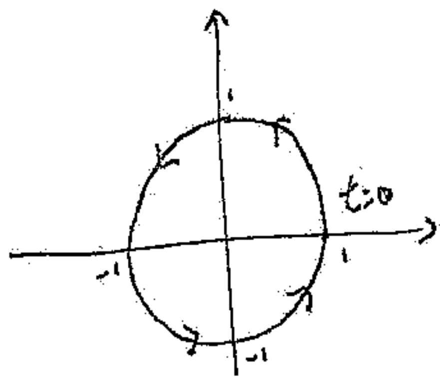
$= t + 2\ln|t-1| - \frac{1}{t-1} + C$.

$\xrightarrow{t = e^x} = e^x + 2\ln|e^x - 1| - \frac{1}{e^x - 1} + C$.

$\square$

5. (a)

$x^2+y^2=1$ $\Rightarrow$ unit circle, counterclockwise parametrization.



(b) $\frac{dy}{dx}$ should be 1.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{-\sin t} = -\cot t = 1$$

$$\Updownarrow$$

$$\tan t = -1$$

$$\Rightarrow t = \frac{3\pi}{4}, \frac{7\pi}{4} \quad (\text{recall } 0 \le t \le 2\pi)$$

$\therefore$ $t = \frac{3\pi}{4}$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$t = \frac{7\pi}{4}$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$