Example 4. Find the Maclaurin series for the function

\[ f(x) = \frac{1}{\sqrt{4-x}} \]

and its radius of convergence.
Example 4. Find the Maclaurin series for the function

\[ f(x) = \frac{1}{\sqrt{4 - x}} \]  
and its radius of convergence.

Solution: Rewrite \( f(x) \) in a form so that we can use the binomial series:

\[
\frac{1}{\sqrt{4 - x}} = \frac{1}{\sqrt{4(1 - \frac{x}{4})}} = \frac{1}{2\sqrt{1 - \frac{x}{4}}} = \frac{1}{2} \left(1 - \frac{x}{4}\right)^{-\frac{1}{2}}. \tag{17}
\]
Taylor and Maclaurin series

Thus, the Maclaurin series is

\[
\frac{1}{2} \left[ 1 + \sum_{n=1}^{\infty} \left( -\frac{1}{2} \right) \cdots \left( -\frac{1}{2} - n + 1 \right) \left( -\frac{x}{4} \right)^n \right].
\]

(18)

This uses binomial series for \((1 - \frac{x}{4})^{-\frac{1}{2}}\). Thus \(|\frac{x}{4}| < 1\), i.e. radius of convergence \(R = 4\).

There is an important table of Taylor (Maclaurin) series on page 768 of chapter 11.10.
Taylor and Maclaurin series

The partial sum up to the term $\frac{f^{(N)}(a)}{N!}(x - a)^N$ is called Taylor polynomial $T_N(x)$, i.e.

$$T_N(x) := \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x - a)^n \quad (19)$$

The difference between $f(x)$ and $T_N(x)$ is called the remainder $R_N(x)$.

$$R_N(x) := f(x) - T_N(x).$$
When is the function \( f(x) \) equal to its Taylor series?

**Theorem 1** If the remainder \( R_n(x) \to 0 \), then \( f(x) \) is equal to its Taylor series.
Theorem 2: (Taylor's inequality) If $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1} \quad \text{for} \quad |x - a| \leq d.$$

We can use this inequality to prove $R_n(x) \to 0$. 

Chapter 11: Sequences and Series, Section 11.8 Power series
Example 5. Prove

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \]

Solution: In Example 1, we have computed the Maclaurin series of the function \( f(x) = e^x \) and its radius of convergence.

The Maclaurin series for \( f \) at 0 is

\[ \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \]
Taylor and Maclaurin series

\[ |R_n(x)| \leq \frac{M}{(n+1)!}|x|^{n+1} \leq \frac{e^d}{(n+1)!}|x|^{n+1} \text{ for } |x| \leq d. \]

And

\[ \lim_{n \to \infty} \frac{e^d}{(n+1)!}|x|^{n+1} = 0. \]

Thus \(|R_n(x)| \to 0\) for all \(x\). By Theorem 1,

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \]
Taylor and Maclaurin series

A similar idea works for $\sin x$ and $\cos x$. Thus we can show the Taylor series of $\sin x$ equals $\sin x$. Also, the Taylor series of $\cos x$ equals $\cos x$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad (20)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad (21)$$
To derive each formula on page 768 of Chapter 11.10, we need 2 steps:

1. Compute $f^{(n)}(a)$, thus to derive the Taylor series of the target function. (This is what we learned on last Friday.)

2. Prove the remainder $R_N(x)$ tends to 0, and thus get the target function equals its Taylor series. (This is what we learn today.)
\[ R_N(x) := f(x) - T_N(x) \]

\[ \lim_{N \to \infty} R_N(x) = f(x) - \lim_{N \to \infty} T_N(x) \]

\[ = f(x) - \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \]

\[ \Downarrow \]

Taylor series

\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \]

\[ \iff \]

\[ \lim_{N \to \infty} R_N(x) = 0 \]

(equivalent)

E.g. for \( \sin x \).

Taylor series is \[ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \]

Is it true that \[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \] ? True if \[ \lim_{N \to \infty} R_N(x) = 0 \]
\[ |R_n(x)| \leq \frac{M}{(N+1)!} (x-a)^{N+1} \]

\[ \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \text{ is the Taylor series of } f(x) \text{ at } a. \]

\[ \left| \frac{f^{(N+1)}(a)}{(N+1)!} (x-a)^{N+1} \right| \leq \frac{M}{(N+1)!} (x-a)^{N+1} \]

Since we want to prove

\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \]

\[ = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!} (x-a)^n + \sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \]

\[ = T_N(x) + R_N(x) \]
The leading order term of $R_N(x)$ is given by

$$\frac{f^{(N+1)}(a)}{(N+1)!} (x-a)^{N+1}$$
\[ |R_n(x)| \leq \frac{M}{(n+1)!} |x|^{n+1} \quad \text{if} \quad |f^{(n+1)}(x)| \leq M \quad \text{for} \quad |x| \leq d \]

Since \( f^{(n+1)}(x) = e^x \), so on \( |x| \leq d \)

\[ |e^x| \leq e^d = M \]

\[ \Rightarrow \quad |R_n(x)| \leq \frac{e^d}{(n+1)!} |x|^{n+1} \quad \text{on} \quad |x| \leq d. \]
Fact \[ \lim_{n \to \infty} \frac{x^n}{n!} = 0 \quad \text{for all } x \]

\[ \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (\text{Taylor series of } e^x \text{ at } 0) \text{ has} \]

radius of convergence = \( \infty \). (by ratio test).

In other words, for all \( x \), \[ \sum_{n=0}^{\infty} \frac{x^n}{n!} \]
converges.

Thus by divergence test, \[ \lim_{n \to \infty} \frac{x^n}{n!} = 0. \]
1. A polynomial \[ \sum_{n=0}^{N} C_n x^n \]'s Taylor series equals itself.

   e.g. \[ x^2 + 3x + 2 \]'s Taylor series at 0 is \[ x^2 + 3x + 2 \].

2. \[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \] at \( a=0 \).

The Taylor series of \( e^x \) at \( a=3 \) is totally different.

It takes the form \[ e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!} \cdot (x-3)^n \]
Applications of Taylor series

- Approximating functions by polynomials
Applications of Taylor series

■ Applications to physics