Strategy for testing series

List of tests we've learned:

1. Divergence test

2. Integral test

3. Comparison test (including limit comparison test and the modified limit comparison test)

4. Alternating series test

5. Absolute convergence test

6. Ratio test

7. Root test

Summary of Convergence Tests for Series

Test	Series	Convergence or Divergence	Comments
n^{th} term test (or the zero test)	$\sum a_n$	Diverges if $\lim_{n\to\infty} a_n \neq 0$	Inconclusive if $\lim_{n\to\infty} a_n = 0$.
Geometric series	$\sum_{n=0}^{\infty} ax^n \left(\text{or} \sum_{n=1}^{\infty} ax^{n-1}\right)$	Converges to $\frac{a}{1-x}$ only if $\|x\| < 1$ Diverges if $\|x\| \geq 1$	Useful for comparison tests if the n^{th} term a_n of a series is similar to ax^n.
p-series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	Converges if $p > 1$ Diverges if $p \leq 1$	Useful for comparison tests if the n^{th} term a_n of a series is similar to $\frac{1}{n^p}$.
Integral	$\sum_{n=c}^{\infty} a_n \quad (c \geq 0)$ $a_n = f(n)$ for all n	Converges if $\int_c^{\infty} f(x)\,dx$ converges Diverges if $\int_c^{\infty} f(x)\,dx$ diverges	The function f obtained from $a_n = f(n)$ must be continuous, positive, decreasing and readily integrable for $x \geq c$.
Comparison	$\sum a_n$ and $\sum b_n$ with $0 \leq a_n \leq b_n$ for all n	$\sum b_n$ converges $\Longrightarrow \sum a_n$ converges $\sum a_n$ diverges $\Longrightarrow \sum b_n$ diverges	The comparison series $\sum b_n$ is often a geometric series or a p-series.
Limit Comparison*	$\sum a_n$ and $\sum b_n$ with $a_n, b_n > 0$ for all n and $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$	$\sum b_n$ converges $\Longrightarrow \sum a_n$ converges $\sum b_n$ diverges $\Longrightarrow \sum a_n$ diverges	The comparison series $\sum b_n$ is often a geometric series or a p-series. To find b_n consider only the terms of a_n that have the greatest effect on the magnitude.
Ratio	$\sum a_n$ with $\lim_{n\to\infty} \frac{\|a_{n+1}\|}{\|a_n\|} = L$	Converges (absolutely) if $L < 1$ Diverges if $L > 1$ or if L is infinite	Inconclusive if $L = 1$. Useful if a_n involves factorials or n^{th} powers.
Root*	$\sum a_n$ with $\lim_{n\to\infty} \sqrt[n]{\|a_n\|} = L$	Converges (absolutely) if $L < 1$ Diverges if $L > 1$ or if L is infinite	Test is inconclusive if $L = 1$. Useful if a_n involves n^{th} powers.
Absolute Value $\sum \|a_n\|$	$\sum a_n$	$\sum \|a_n\|$ converges $\Longrightarrow \sum a_n$ converges	Useful for series containing both positive and negative terms.
Alternating series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \quad (a_n > 0)$	Converges if $0 < a_{n+1} < a_n$ for all n and $\lim_{n\to\infty} a_n = 0$	Applicable only to series with alternating terms.

*The Root and Limit Comparison tests are not included in the current textbook used in Calculus classes at Bates College.

Strategy for testing series

List of standard series:

1. p-series (including harmonic series) converges if and only if $p > 1$
2. geometric series convergence if and only if $|r| < 1$

Strategy for testing series

- Example. (problem 19 on page 746.)

$$\sum_{n=1}^{\infty}(-1)^n \frac{\ln n}{\sqrt{n}}$$

Solution: Alternating series test

Strategy for testing series

- Example. (problem 20 on page 746)

$$\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}-1}{k(\sqrt{k}+1)}$$

Solution: Limit comparison test

Strategy for testing series

- Example. (problem 21 on page 746)

$$\sum_{n=1}^{\infty}(-1)^n \cos\frac{1}{n^2}$$

Solution: Divergence test

Strategy for testing series

- Example. (problem 22 on page 746)

$$\sum_{k=1}^{\infty} \frac{1}{2+\sin k}$$

Solution: Divergence test

Strategy for testing series

- Example. (problem 23 on page 746)

$$\sum_{n=1}^{\infty} \tan(\frac{1}{n})$$

Solution: Limit comparison test

Strategy for testing series

- Example. (problem 24 on page 746)

$$\sum_{n=1}^{\infty} n \sin(\frac{1}{n})$$

Solution: Divergence test

Strategy for testing series

- Example. (problem 25 on page 746)

$$\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$$

Solution: Ratio test

Strategy for testing series

- Example. (problem 26 on page 746)

$$\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$$

Solution: Root test

Strategy for testing series

- Example. (problem 27 on page 746)

$$\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$$

Solution: Compare with $\frac{k \ln k}{k^3}$