Taylor and Maclaurin series

Definition

The power series

\[ f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots \]

is called Taylor series of function \( f(x) \) at point \( a \).
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If \( a = 0 \), we call

\[
f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots.
\]

Maclaurin series

This case arises frequently enough that it is given the special name.
Example 1. Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

Solution:

$$f(x) = e^x$$
$$f'(x) = e^x$$
$$f''(x) = e^x$$
$$f^{(n)}(x) = e^x$$ for all $n$.

Thus $f^{(n)}(0) = e^0 = 1$ for all $n$.

The Taylor series for $f$ at 0 is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
To find the radius of convergence we let $a_n = \frac{x^n}{n!}$. Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{|x|}{n + 1} \to 0 < 1.$$ (10)

for all $x$. Thus by the ratio test, the series converges for all $x$ and the radius of convergence is $R = \infty$. 


Example 3. Find the Maclaurin series for \( \sin x \) and \( \cos x \).

Solution:

\[
\sin x \bigg|_{x=0} = 0; \quad (\sin x)' \bigg|_{x=0} = 1; \quad (\sin x)'' \bigg|_{x=0} = 0; \quad (\sin x)''' \bigg|_{x=0} = -1.
\]

Since the derivatives repeat in a cycle of four, we see that

\[
\sin^{(4)} x \bigg|_{x=0} = 0; \quad \sin^{(5)} x \bigg|_{x=0} = 1; \quad \sin^{(6)} x \bigg|_{x=0} = 0; \quad \sin^{(7)} x \bigg|_{x=0} = -1; \ldots
\]

Hence the Maclaurin series is

\[
x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
\]
In fact, we can show by estimating the remainder

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots 
\]  

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For \( \cos x \), we can proceed directly as above. But it is easier to differentiate the Maclaurin series of \( \sin x \) term by term.
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\[
\cos x = \frac{d}{dx} (\sin x) = \frac{d}{dx} \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right)
\]

\[
= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots 
\]

\[
= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots 
\]

Remark: The Maclaurin series of \(\sin x\) consists of only odd powers, and that of \(\cos x\) consists of only even powers.
Let $k$ be any real number and $|x| < 1$.

How to expand $(1 + x)^k$ in Maclaurin series? ($k$ is in general not an integer.)

\[
(1 + x)^k \big|_{x=0} = 1; \quad ((1 + x)^k)' \big|_{x=0} = k; \quad ((1 + x)^k)'' \big|_{x=0} = k(k - 1); \quad \text{(13)}
\]

\[
((1 + x)^k)''' \big|_{x=0} = k(k - 1)(k - 2); \quad \cdots \quad \text{(14)}
\]
Thus the Maclaurin series is

\[ 1 + \sum_{n=1}^{\infty} \frac{k(k - 1)(k - 2) \cdots (k - n + 1)}{n!} x^n. \]  

This is called a binomial series. We also need to find the radius of convergence.
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Let’s determine the radius of convergence by ratio test.

\[
|\frac{a_{n+1}}{a_n}| = \frac{|k - n|}{(n + 1)}|x| = \frac{|1 - \frac{k}{n}|}{1 + \frac{1}{n}}|x| \to |x|,
\]

(16)

as \(n \to \infty\). Thus the radius of convergence is \(R = 1\).