General Principle

Choose $f(x)$ based on which of these comes first

- I–Inverse functions (e.g. $\arcsin x$, $\arccos x$, etc.)
- L–Logarithmic functions (e.g. $\log x$, $\log_2 x$, $\log_{10} x$ etc.)
- A–Algebraic functions (e.g. x^3, x^9, etc.)
- T–Trig functions (e.g. $\sin x$, $\cos x$, etc.)
- E–Exponential functions (e.g. $e^x, 2^x, 3^x$, etc.)

ILATE

Example 2

Sometime we need to use a trick.

- Integrate $\int \sin x \cdot e^x dx$.

  According to ILATE, we choose $f(x) = \sin x$.

$$\begin{aligned} &\int \sin x \cdot e^x dx \\ =&\sin x \cdot e^x - \int \cos x \cdot e^x dx. \end{aligned} \tag{3}$$

It seems it hasn't improved anything; $\int \cos x \cdot e^x dx$ is as difficult as $\int \sin x \cdot e^x dx$. But we derive a recursive formula if we do integration by parts once more.

Example 2

$$
\begin{aligned}
&= \sin x \cdot e^x - \int \cos x \cdot e^x dx \\
&= \sin x \cdot e^x - \left(\cos x \cdot e^x - \int (-\sin x) \cdot e^x dx\right) \qquad (4) \\
&= \sin x \cdot e^x - \cos x \cdot e^x - \int \sin x \cdot e^x dx.
\end{aligned}
$$

Thus

$$\int \sin x \cdot e^x dx = \sin x \cdot e^x - \cos x \cdot e^x - \int \sin x \cdot e^x dx. \tag{5}$$

$\Rightarrow$

$$\int \sin x \cdot e^x dx = \frac{1}{2}(\sin x \cdot e^x - \cos x \cdot e^x) + C. \tag{6}$$

Example 3

- Integrate $\int \arctan x dx$.

  We can think of $\arctan x$ as $1 \cdot \arctan x$, and note $1 = x^0$ is an algebraic function. Thus according to ILATE, we choose $f(x) = \arctan x$.

$$\begin{aligned}&\int \arctan x dx \\ =&\arctan x \cdot x - \int x \cdot \frac{1}{1+x^2} dx.\end{aligned} \tag{7}$$

  Substitute $y = x^2$, then $dy = 2xdx$. Thus

$$\int x \cdot \frac{1}{1+x^2} dx = \int \frac{1}{1+y} dy = \ln|1+y| + C. \tag{8}$$

Example 4, 5

Practice problems:

- Example 4. $\int x^3\sqrt{1+x^2}dx$
- Example 5. $\int x^2 \ln x dx$

7.2: Integrals of trigonometrics I

- Example 1. Evaluate $\int \cos^3 x dx$.

$$\int \cos^3 x dx = \int \cos^2 x \cdot \cos x dx$$
$$= \int (1 - \sin^2 x) \cos x dx \quad (9)$$
$$= \int (1 - u^2) du$$

where we substitute $u = \sin x$, $du = \cos x dx$. We use the identity

$$\sin^2 x + \cos^2 x = 1.$$

7.2: Integrals of trigonometrics I

- It is easy to take integral of $(1 - u^2)$.

$$
\begin{aligned}
&= \int (1 - u^2) du \\
&= u - \frac{1}{3} u^3 + C \\
&= \sin x - \frac{1}{3} \sin^3 x + C.
\end{aligned} \tag{10}
$$

7.2: Integrals of trigonometrics I

- Recall useful trigonometric identities:

$$1. \quad \sin^2 x + \cos^2 x = 1,$$

$$2. \quad \sin 2x = 2 \sin x \cos x$$

$$3. \quad \cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

In fact, 2 and 3 follow from the following general summation rule:

$$4. \quad \sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$$

$$5. \quad \cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$$

7.2: Integrals of trigonometrics I

4. and 5 also deduce useful fomula for the product of sine and cosine.

$$6. \quad \sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$7. \quad \sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

7.2: Integrals of trigonometrics I

- Goal: evaluate $\int \sin^m x \cos^n x dx$, where m, n are integers.
- Strategy:
  - Case 1. If n is an odd integer, then substitute $u = \sin x$ and use $\cos^2 x = 1 - \sin^2 x$. It gives

$$\begin{aligned} \int \sin^m x \cos^n x dx &= \int \sin^m x (1 - \sin^2 x)^{\frac{n-1}{2}} \cos x dx \\ &= \int u^m (1 - u^2)^{\frac{n-1}{2}} du. \end{aligned} \tag{11}$$

This is a polynomial in u, since when n is odd, $(n-1)/2$ is an integer. It is easy to integrate! Final step is to substitute back to x.