Simplicial Complexes and Δ-Complexes

This note expands on some of the material on Δ-complexes in §2.1 of Hatcher's book *Algebraic Topology*, from a slightly more classical point of view.

Simplices Given any linearly independent set $V = \{v_0, v_1, \ldots, v_n\}$ of $n+1$ points in $\mathbb{R}^N$, the *n-simplex with vertices* V is the convex hull of V, i. e. the set of all points of the form $t_0v_0 + t_1v_1 + \ldots + t_nv_n$, where $\sum_{i=0}^n t_i = 1$ and $t_i \geq 0$ for all i (see pp. 102–3). Let us call it Δ_V. (We can relax the condition on V slightly; we only need the n vectors $v_1 - v_0, \ldots, v_n - v_0$ to be linearly independent.)

Low-dimensional examples are familiar enough: for $n = 1$ we find the line segment joining v_0 to v_1; for $n = 2$ we find the triangle with vertices v_0, v_1 and v_2; for $n = 3$ we find the tetrahedron with vertices V. The definition makes sense even for $n = 0$; we simply obtain the one-point space $\{v_0\}$ as the 0-simplex with vertex v_0. However, the empty set is not considered a simplex.

Definition 1 The *standard n-simplex* $\Delta^n \subset \mathbb{R}^{n+1}$ has the standard basis $\{e_0, e_1, \ldots, e_n\}$ as its set of vertices, i. e.

$$\Delta^n = \left\{ (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n t_i = 1, t_i \geq 0 \text{ for all } i \right\}.$$

Given $0 \leq i \leq n$, the *i-th face* F_i of Δ^n is the subspace of points that satisfy $t_i = 0$; it is the $(n-1)$-simplex whose vertices are all those of Δ^n except e_i. The *boundary* $\partial\Delta^n$ of Δ^n is the subspace $\bigcup_{i=0}^n F_i$.

For each i, we have the obvious map $\eta_i\colon \Delta^{n-1} \cong F_i \subset \Delta^n$, where $\eta_i(e_j) = e_j$ if $j < i$, but $\eta_i(e_j) = e_{j+1}$ for $j \geq i$.

The linear map $T\colon \mathbb{R}^{n+1} \to \mathbb{R}^N$ given by

$$T(t_0, t_1, \ldots, t_n) = t_0v_0 + t_1v_1 + \ldots + t_nv_n$$

induces the obvious bijection $\sigma_V\colon \Delta^n \to \Delta_V$ (see p. 103), which is a homeomorphism since Δ^n is compact and Δ_V is Hausdorff. (Some books adopt a different choice of standard n-simplex Δ^n, but this observation shows that it does not matter.)

Simplicial complexes in $\mathbb{R}^N$ Let V be a linearly independent set of points in $\mathbb{R}^N$. For each subset $\alpha \subset V$, we have defined the simplex Δ_α. The subspace X of $\mathbb{R}^N$ formed by taking the union of *some* of these simplices is called a (geometric) *simplicial complex*. Its *n-skeleton* $X^n \subset X$ is formed by keeping only the i-simplices for $i \leq n$. Since there is a homeomorphism $(\Delta^n, \partial\Delta^n) \cong (D^n S^{n-1})$, it is clear that X is a finite CW-complex, with one n-cell for each n-simplex. Despite appearances, simplicial complexes include many spaces of interest.

It is easy to see that the space X depends only on which subsets α are used, not on the choice of embedding $V \subset \mathbb{R}^N$. (Indeed, the condition on V may be relaxed somewhat, but this does not concern us.) Clearly, if $\Delta_\alpha \subset X$ and $\beta \subset \alpha$, then $\Delta_\beta \subset \Delta_\alpha \subset X$. This leads to the following combinatorial description (see p. 107). We also wish to allow V to be infinite, and to free ourselves of reliance on $\mathbb{R}^N$, which may suggest an inappropriate topology on X.

DEFINITION 2 An *abstract simplicial complex* K consists of a set V, whose elements are called *vertices*, and a collection S of finite non-empty subsets of V that satisfies the axioms:

(i) For each $v \in V$, the singleton $\{v\} \in S$;
(ii) If $\alpha \in S$ and $\beta \subset \alpha$ is non-empty, then $\beta \in S$.

A member $\alpha \in S$ consisting of $n+1$ elements is called an *n-simplex* of K.

As before, the *n-skeleton* K^n of K is obtained by keeping only the i-simplices for $i \le n$.

One can construct the *geometric realization* $|K|$ of K. For each simplex α, take a copy Δ_α of Δ^n, where n is the dimension of α, and identify the faces appropriately to construct the space $|K|$ as a quotient of the disjoint union $\coprod_\alpha \Delta_\alpha$.

We do not pursue this approach any further. We note that the homeomorphism $\sigma_\alpha\colon \Delta^n \cong \Delta_\alpha$ is not uniquely defined until we choose an ordering of the set α. For future purposes, we need a preferred σ_α, which leads to the following concept.

Ordered simplicial complexes

DEFINITION 3 An *ordered abstract simplicial complex* K is an abstract simplicial complex as above, in which the set V of vertices is ordered. We write K_n for the set of n-simplices of K. Then for $n > 0$ and $0 \le i \le n$, we define the *face operator* $d_i\colon K_n \to K_{n-1}$ on $\alpha = \{v_0, v_1, \dots, v_n\}$ by omitting v_i, where the vertices $v_0 < v_1 < \dots < v_n$ are arranged in order. (Again, some books get by with something less than a total ordering on V, but the present approach is simplest.)

LEMMA 4 *We have*

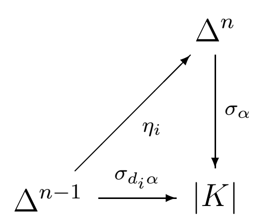
$$d_i \circ d_j = d_j \circ d_{i+1}\colon K_n \longrightarrow K_{n-2} \quad \textit{whenever } n \ge 2 \textit{ and } i \ge j. \tag{5}$$

Proof To compute $d_i d_j \alpha$, we first delete v_j. Then v_{i+1} moves left one place to position i, where it is deleted by d_i. On the other hand, if we delete v_{i+1} first, v_j does not move, and is deleted by d_j. □

Geometric realization We construct the geometric realization $|K|$ of an ordered abstract simplicial complex in a way that generalizes the finite case in $\mathbb{R}^N$. It will have a map

$$\sigma_\alpha\colon \Delta^n \longrightarrow |K| \tag{6}$$

for each simplex α of K, where n is the dimension of α. To ensure that the faces of $\sigma_\alpha(\Delta^n)$ really are what they should be, we require for each i the commutative triangle



(7)

The construction is inductive. It begins with the 0-skeleton $|K^0| = K_0$, treated as a discrete space, where for each $\alpha \in K_0$, $\sigma_\alpha \colon \Delta^0 \to |K^0| = K_0$ has image α.

Suppose we have already constructed the $(n-1)$-skeleton $|K^{n-1}|$, with maps σ_α for all simplices α of dimension less than or equal to $n-1$, satisfying condition (7). We construct $|K^n|$ from it by *attaching n-cells*, one for each $\alpha \in K_n$, by the pushout square

$$\begin{CD}
\coprod_\alpha \partial\Delta^n @>{f}>> |K^{n-1}| \\
@V{\subset}VV @VV{\subset}V \\
\coprod_\alpha \Delta^n @>{g}>> |K^n|
\end{CD}$$

On the α-th copy of Δ^n, the resulting g will provide the characteristic map

$$\sigma_\alpha \colon (\Delta^n, \partial\Delta^n) \longrightarrow (|K^n|, |K^{n-1}|).$$

We need to construct the attaching map f. We write $f_\alpha \colon \partial\Delta^n \to |K^{n-1}|$ for its restriction to the α-th copy of $\partial\Delta^n$. To satisfy diagram (7), f_α must be given on each face F_i of Δ^n as

$$F_i \cong \Delta^{n-1} \xrightarrow{\sigma_{d_i\alpha}} |K^{n-1}|.$$

We must check that f_α is well defined on $F_i \cap F_j$ whenever $i < j$ (assuming $n \geq 2$; if $n = 1$ there is nothing to prove). On $F_i \cap F_j$, considered as a subspace of F_i, it is given by the commutative diagram

$$\begin{array}{ccccc}
F_i \cap F_j & \xrightarrow{\cong} & F'_{j-1} & \xrightarrow{\cong} & \Delta^{n-2} \\
\downarrow{\scriptstyle\subset} & & \downarrow{\scriptstyle\subset} & \swarrow{\scriptstyle\eta_{j-1}} & \downarrow{\scriptstyle\sigma_{d_{j-1}d_i\alpha}} \\
F_i & \xrightarrow{\cong} & \Delta^{n-1} & \xrightarrow{\sigma_{d_i\alpha}} & |K^{n-1}|
\end{array}$$

where F'_k denotes the k-th face of Δ^{n-1}. We note that because $i \leq j-1$, the homeomorphism $\Delta^{n-1} \cong F_i$ carries F'_{j-1} (not F'_j) to $F_i \cap F_j$.

If instead we regard $F_i \cap F_j$ as a subspace of F_j, we obtain the commutative diagram

$$\begin{array}{ccccc}
F_i \cap F_j & \xrightarrow{\cong} & F'_i & \xrightarrow{\cong} & \Delta^{n-2} \\
\downarrow{\scriptstyle\subset} & & \downarrow{\scriptstyle\subset} & \swarrow{\scriptstyle\eta_i} & \downarrow{\scriptstyle\sigma_{d_id_j\alpha}} \\
F_j & \xrightarrow{\cong} & \Delta^{n-1} & \xrightarrow{\sigma_{d_j\alpha}} & |K^{n-1}|
\end{array}$$

Since both homeomorphisms $F_i \cap F_j \cong \Delta^{n-2}$ are the standard order-preserving map, Lemma 4 shows that the two maps agree on $F_i \cap F_j$. This completes the induction step.

To make $|K|$ a CW-complex, we topologize it as a quotient space of $\coprod_\alpha \Delta^n$.

A *triangulation* of any space X is a homeomorphism $|K| \cong X$ for some simplicial complex K.

Δ-complexes In the above geometric realization, we observe that we nowhere used the fact that each α is a set of vertices. We therefore generalize.

DEFINITION 8 An *abstract* Δ*-complex* K is a sequence of sets K_0, K_1, K_2, ..., with face operators $d_i\colon K_n \to K_{n-1}$ for $n > 0$ and $0 \le i \le n$ that satisfy equation (5).

We construct its geometric realization $|K|$ exactly as above; the result is a Δ-complex as described on p. 104. The only fact we needed was Lemma 4, which we took here as an axiom. This does not enlarge the class of spaces $|K|$, as Exercise 23 shows how to triangulate $|K|$ as a simplicial complex for any Δ-complex K. However, Δ-complexes are often convenient to work with in practice, as they typically contain many fewer simplices than any simplicial triangulation of $|K|$.